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## A FINITE ELEMENT METHOD FOR A CURLCURL-GRADDIV EIGENVALUE INTERFACE PROBLEM\*

HUOYUAN DUAN<sup>†</sup>, PING LIN<sup>‡</sup>, AND ROGER C. E. TAN<sup>§</sup>

**Abstract.** In this paper we propose and study a finite element method for a curlcurl-graddiv eigenvalue interface problem. Its solution may be of piecewise non- $H^1$ . We would like to approximate such a solution in an  $H^1$ -conforming finite element space. With the discretizations of both curl and div operators of the underlying eigenvalue problem in two finite element spaces, the proposed method is essentially a standard  $H^1$ -conforming element method, up to element bubbles which can be statically eliminated at element levels. We first analyze the proposed method for the related source interface problem by establishing the stability and the error bounds. We then analyze the underlying eigenvalue interface problem, and we obtain the error bounds  $\mathcal{O}(h^{2r_0})$  for eigenvalues which correspond to eigenfunctions in  $\prod_{j=1}^J (H^r(\Omega_j))^3 \hookrightarrow (H^{r_0}(\Omega))^3$  space, where the piecewise regularity  $r$  and the global regularity  $r_0$  may belong to the most interesting interval  $[0, 1]$ .

**Key words.** generalized Maxwell eigenvalue problem of curlcurl-graddiv operator,  $H^1$ -conforming finite element method, piecewise non- $H^1$ -space solution, error estimates, spectral correctness

**AMS subject classification.** 65N30

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**1. Introduction.** In this paper we study the finite element method for solving a generalized Maxwell eigenvalue interface problem, where the governing operator is the curlcurl-graddiv operator, arising from computational electromagnetism. The finite element method features that the solution is sought in an  $H^1$ -conforming finite element space. Given a simply connected domain  $\Omega \subset \mathbb{R}^3$ , with a connected boundary  $\Gamma$ , let  $\mathbf{n}$  denote the outward unit normal vector to  $\Gamma$ . Let  $\mu, \varepsilon$  represent the matrices of the physical properties such as permeability and permittivity of the materials occupying  $\Omega$ . The generalized Maxwell eigenvalue interface problem of curlcurl-graddiv operator is defined as follows: Find  $(\omega^2, \mathbf{u} \neq \mathbf{0})$  such that

$$(1.1) \quad \text{curl} \mu^{-1} \text{curl} \mathbf{u} - \varepsilon \nabla \text{div} \varepsilon \mathbf{u} = \omega^2 \varepsilon \mathbf{u} \quad \text{in } \Omega, \quad \mathbf{u} \times \mathbf{n} = \mathbf{0}, \quad \text{div} \varepsilon \mathbf{u} = 0 \quad \text{on } \Gamma.$$

A closely related model in computational electromagnetism is

$$(1.2) \quad \text{curl} \mu^{-1} \text{curl} \mathbf{u} = \omega^2 \varepsilon \mathbf{u}, \quad \text{div} \varepsilon \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

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Introduce Hilbert spaces  $H(\mathbf{curl}; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \mathbf{curl} \mathbf{v} \in (L^2(\Omega))^3\}$ ,  $H_0(\mathbf{curl}; \Omega) = \{\mathbf{v} \in H(\mathbf{curl}; \Omega) : \mathbf{v} \times \mathbf{n}|_\Gamma = \mathbf{0}\}$ ,  $H(\operatorname{div} \varepsilon; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \operatorname{div} \varepsilon \mathbf{v} \in L^2(\Omega)\}$ ,  $H(\operatorname{div}^0 \varepsilon; \Omega) = \{\mathbf{v} \in H(\operatorname{div} \varepsilon; \Omega) : \operatorname{div} \varepsilon \mathbf{v} = 0\}$ ,  $H_0(\operatorname{div} \varepsilon; \Omega) = \{\mathbf{v} \in H(\operatorname{div} \varepsilon; \Omega) : \varepsilon \mathbf{v} \cdot \mathbf{n}|_\Gamma = 0\}$ . With the notation  $(\cdot, \cdot)$  for the  $L^2$  inner product, the classical variational statement of (1.1) is to find  $(\omega^2, \mathbf{u} \neq \mathbf{0}) \in \mathbb{R} \times H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  such that

$$(1.3) \quad (\mu^{-1} \mathbf{curl} \mathbf{u}, \mathbf{curl} \mathbf{v}) + (\operatorname{div} \varepsilon \mathbf{u}, \operatorname{div} \varepsilon \mathbf{v}) = \omega^2 (\varepsilon \mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega).$$

Problem (1.3) is also well-known as the so-called plain regularization method of (1.2), accounting for the divergence-free constraint [29, 30, 32, 44]. We refer to section 1 in [30] for the connection of (1.1) and (1.2) and section 7 in [30] for the variational statement (1.3) of (1.1).

As a second-order elliptic eigenproblem, quite like the Laplacian eigenproblem, the classical nodal-continuous or  $H^1$ -conforming finite element method is most desirable for numerically solving problem (1.3). In fact, the  $H^1$ -conforming finite element method is appropriate whenever the solution/eigenfunction belongs to  $H^1$ -space and  $\varepsilon$  is smooth. Note that the conforming finite element space of piecewise polynomials for (1.3) is necessarily  $H^1$ -conforming (see [47, 46, 17]).

However, although the curl operator and the div operator closely relate to the gradient operator  $\nabla$ , with  $\mathbf{curl} \mathbf{u} = \nabla \times \mathbf{u}$  and  $\operatorname{div} \varepsilon \mathbf{u} = \nabla \cdot \varepsilon \mathbf{u}$ , the resultant eigenfunctions may have some nonsmooth ones, which belong not to  $H^1$ -space but to fractional-order Hilbert spaces  $H^r$ , where  $0 \leq r < 1$ . This situation with very low regularity eigenfunctions is not rare. As a matter of fact, non- $H^1$ -space solutions are commonplace in electromagnetism. A main cause is due to the reentrant corners and edges along the domain boundary  $\Gamma$  and across the interfaces of different materials occupying the domain  $\Omega$  (see [30, 34, 29, 33, 4, 14] for details). For a non- $H^1$ -space solution, it has been widely recognized that the  $H^1$ -conforming finite element solution of the eigenproblem (1.3) cannot correctly converge (e.g., see [17, 45, 49]). It turns out that the bilinear form in (1.3) accounts for this failure. To obtain a correctly convergent  $H^1$ -conforming finite element solution, the only option is how to modify/discretize the bilinear form in (1.3) in the finite element method. Although for many years there have been many attempts on how to use the  $H^1$ -conforming finite element method to correctly approximate a non- $H^1$ -space solution, it is only during the last decade that we have seen a few successful methods. For (1.2), there are the weighted method [31, 21, 48, 25], the  $H^{-\alpha}$ -method for some  $1/2 < \alpha \leq 1$  [18, 11, 15], and the  $L^2$  projection method [36, 35]. We should also mention the earlier work on the singular complement method [5, 6, 7, 8, 9], which deals with Maxwell equations in a domain with reentrant corners as well.

To motivate the study of this paper, we first review the several mentioned methods.

In the case of solving problem (1.1), it is not clear whether both the weighted method and the  $H^{-\alpha}$ -method can be applied. This is because the main idea for both is to introduce a weaker norm (i.e., weighted  $L^2$ -norm or  $H^{-\alpha}$ -norm) than the standard  $L^2$ -norm for measuring the div operator. The weighted  $L^2$  inner product or using the  $H^{-\alpha}$  inner product for the div term in (1.1) may introduce a different problem from (1.1). In the case of the Maxwell eigenvalue problem (1.2), the weighted method and the  $H^{-\alpha}$ -method can be well-defined by a least-squares approach which deals with the divergence-free constraint as an independent first-order equation. If we consider only problem (1.2), both the weighted method and the  $H^{-\alpha}$ -method may be suitable with the use of the  $H^1$ -conforming element.

On the other hand, the  $L^2$  projection method can be used for solving (1.1) as well as (1.2), because the main idea of this method is to discretize the div operator and the curl operator in additional finite element spaces. The discretizations of the div and curl operators are a type of  $L^2$  finite element projections, mimicking the classical distributional definitions of the div operator and the curl operator [36, 39, 35, 37]. This paper is a continuation of our previous works. These works are reviewed to enforce the motivation of the study in this paper.

In [36], we study the source problem of (1.2) in homogeneous media. We adopt local  $L^2$  projections for both curl and div operators and the Maxwell solution is required to lie in  $H^r$  for  $r > 1/2$ . We do not study the interface problem. Besides element bubbles, higher-order face element bubbles are also used to enrich the finite element space of the solution. Note that face element bubbles cannot be locally eliminated.

In [35], we study the source problem of (1.2) in discontinuous media, adopting local  $L^2$  projection for the curl operator while mass-lumping  $L^2$  projection for the div operator. Again, the regularity  $r > 1/2$  of the Maxwell solution is necessary in the error analysis and higher-order face and element bubbles are used.

In [39], we study the source problem of (1.1) but focus on the homogeneous media in two dimensions. The method admits the regularity in  $H^r$  for any  $r \in [0, 1]$ . The method is a standard  $H^1$ -conforming linear element method, up to element bubbles. But it is not known whether the argument of the analysis of the stability can be valid if considering the direct generalization of the method to three-dimensional problems. In this work, the case with  $\varepsilon$  being discontinuous is not studied.

All these works do not consider the associated eigenvalue problem. In [37], we study the eigenvalue problem (1.1) in three dimensions, but in homogeneous media, using local  $L^2$  projections for both div and curl operators. The method still requires the eigenfunctions to lie in  $H^r$  for some  $r > 1/2$  and involves the enrichment with higher-order face element bubbles as well as element bubbles. The discontinuous media are not studied. It is left as unknown whether the argument of the stability analysis therein is applicable for higher-order elements.

So, it remains highly desirable to develop a general globally  $H^1$ -conforming finite element method and a general theory for the generalized Maxwell eigenvalue interface problem (1.1) (including the related indefinite source interface problem). Based on the above review, we would require that the  $H^1$ -conforming method for (1.1) can accommodate the following mathematical and numerical considerations:

- The method is globally  $H^1$ -conforming, and, up to element bubbles, the method is essentially a standard  $H^1$ -conforming finite element method (e.g., linear element method), even if the coefficients  $\mu$  and  $\varepsilon$  are discontinuous and anisotropic inhomogeneous. As is well-known, a standard static elimination of the element bubbles leads to a standard finite element method. In the case of the linear element method, this is relevant for three-dimensional problems, with only  $4 \times 3 = 12$  degrees of freedom of nodal-value type on each tetrahedron element.

- Very low piecewise regularity is allowed, say,  $\mathbf{u} \in \prod_{j=1}^J (H^r(\Omega_j))^3 \hookrightarrow (H^{r_0}(\Omega))^3$ , where  $0 \leq r, r_0 \leq 1/2$ , as well as  $1/2 < r, r_0 \leq 1$  are allowed. Error bounds  $\mathcal{O}(h^{2r_0})$  for eigenvalues and  $\mathcal{O}(h^{r_0})$  for eigenfunctions can be obtained. Here  $r_0$  stands for the global regularity exponent and  $r$  the piecewise regularity exponent, and they coincide (i.e.,  $r_0 = r$ ) when the partition of the domain is trivial with global continuous material coefficients. As is well-known, although the global regularity of most interface problems is very low, the piecewise regularity may be higher. By contrast, however, in

the case of the Maxwell interface problem, the piecewise regularity may still possibly be very low. There exist some examples with a piecewise  $r$  being close to zero [34]. Whenever eigenfunctions are more regular and higher-order elements are used, the theory is straightforwardly valid.

So far, to the authors' knowledge, none of the existing methods and theory in the literature can fulfill the above mathematical and numerical considerations for (1.1) as well as (1.2).

In this paper, we shall study the eigenvalue interface problem (1.1) and the related source interface problem. We propose a new  $H^1$ -conforming finite element method, where new  $L^2$  projections are introduced to discretize the curl and the div operators (see section 3). With the newly proposed method, the above several aspects of mathematical and numerical considerations are all fulfilled. We should particularly point out a novel technique in the stability analysis, i.e., the so-called bounded co-chain projection in [3] plays a critical role. It is the use of this projection that the argument in analyzing the stability of the proposed method covers higher-order elements in three dimensions (see Theorem 4.3). Meanwhile, the spectral correctness of the proposed method is shown in this paper mainly from the argument of collective compactness (see Theorem 6.2). These are new in the context of the  $L^2$  projection method.

The rest of this paper is arranged as follows. In section 2, Hilbert spaces and norms, notations, and the relationship of the eigenproblem and the source problem are reviewed. The finite element method is defined in section 3. We develop the Fortin interpolation, the Inf-Sup condition, and the dual Fortin interpolation in section 4, and we shall use these Fortin interpolations in section 5 to establish the coercivity and the error estimates of the finite element method of the source problem. Error estimates for the underlying eigenvalue interface problem are developed in section 6. Concluding remarks are made in the last section.

**2. Preliminaries.** In this section, we review the coefficient matrices  $\mu$  and  $\varepsilon$ , the curl and div Hilbert spaces, and the relationship between the eigenproblem and the corresponding source problem. Throughout the paper, we shall assume that  $\mu = (\mu_{ij}), \varepsilon = (\varepsilon_{ij})$  are symmetric in  $\mathbb{R}^{3 \times 3}$ , uniformly coercive, and in  $(L^\infty(\Omega))^{3 \times 3}$ :

$$\varepsilon_{ij} = \varepsilon_{ji}, \mu_{ij} = \mu_{ji} \quad C^{-1}|\xi|^2 \geq \xi \cdot \varepsilon \xi, \xi \cdot \mu \xi \geq C|\xi|^2 \quad \text{a.e. } \Omega, \forall \xi \in \mathbb{R}^3.$$

Let  $v$  represent any of  $\mu, \mu^{-1}, \varepsilon, \varepsilon^{-1}$ . We introduce the  $v$ -weighted  $L^2$  inner products as follows:

$$(\mathbf{u}, \mathbf{v})_v = (v\mathbf{u}, \mathbf{v});$$

the induced norm is  $\|\mathbf{v}\|_{0,v}$ , which is equivalent to the  $L^2$ -norm  $\|\cdot\|_0$ . Since the materials occupying  $\Omega$  may have different physical properties (e.g., permeability and permittivity in electromagnetic phenomena) in different subregions of  $\Omega$ , we allow  $\mu$  and  $\varepsilon$  to be discontinuous. In other words, there exists a partition  $\mathcal{P} = \{\Omega_j : 1 \leq j \leq J\}$  of  $\Omega$  such that  $\mu$  and  $\varepsilon$  are discontinuous across the interfaces among  $\Omega_j$ ,  $1 \leq j \leq J$ . We shall also assume that  $\mu|_{\Omega_j}$  and  $\varepsilon|_{\Omega_j}$  belong to  $(W^{1,\infty}(\Omega_j))^{3 \times 3}$ ,  $1 \leq j \leq J$ . Introduce the usual Hilbert spaces [1]:  $H^1(\Omega) = \{q \in L^2(\Omega) : \nabla q \in (L^2(\Omega))^3\}$ ,  $H_0^1(\Omega) = \{q \in H^1(\Omega) : q|_\Gamma = 0\}$ ,  $H^1(\Omega)/\mathbb{R} = \{q \in H^1(\Omega) : \int_\Omega q = 0\}$ . Let  $H^1(\Omega)$ ,  $H_0^1(\Omega)$ , and  $H^1(\Omega)/\mathbb{R}$  be equipped with norm  $\|q\|_1^2 = \|q\|_0^2 + \|\nabla q\|_0^2$  and seminorm  $|q|_1 = \|\nabla q\|_0$ . We also need Hilbert space  $H^s(\Omega)$  with norm  $\|q\|_s$  for  $s \in \mathbb{R}$ , where  $H^0(\Omega) = L^2(\Omega)$ . In addition, for div and curl Hilbert spaces  $H(\mathbf{curl}; \Omega)$  and  $H(\mathbf{div}; \Omega)$  as introduced in the previous section, the norm for  $H(\mathbf{curl}; \Omega)$  is  $\|\mathbf{v}\|_{0,\mathbf{curl}}^2 = \|\mathbf{v}\|_0^2 + \|\mathbf{curl} \mathbf{v}\|_0^2$  and the norm for  $H(\mathbf{div}; \Omega)$  is  $\|\mathbf{v}\|_{0,\mathbf{div}\varepsilon}^2 = \|\mathbf{v}\|_0^2 +$

$\|\operatorname{div} \varepsilon \mathbf{v}\|_0^2$ . The Hilbert space  $H(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  is equipped with the norm  $\|\mathbf{v}\|_{0; \mathbf{curl}; \operatorname{div} \varepsilon}^2 = \|\mathbf{v}\|_0^2 + \|\mathbf{curl} \mathbf{v}\|_0^2 + \|\operatorname{div} \varepsilon \mathbf{v}\|_0^2$ . As a result of the following proposition, the Hilbert space  $H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  or  $H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div} \varepsilon; \Omega)$  can be equipped with the norm  $\|\mathbf{v}\|_{0; \mathbf{curl}; \operatorname{div} \varepsilon}^2 := \|\mathbf{curl} \mathbf{v}\|_0^2 + \|\operatorname{div} \varepsilon \mathbf{v}\|_0^2$ , which is equivalent to the norm  $\|\mathbf{v}\|_{0; \mathbf{curl}; \operatorname{div} \varepsilon}$ . Using  $\mu$  to replace  $\varepsilon$ , we can similarly define  $H(\operatorname{div} \mu; \Omega)$ ,  $H(\operatorname{div}^0 \mu; \Omega)$ ,  $H_0(\operatorname{div} \mu; \Omega)$  and  $H_0(\operatorname{div}^0 \mu; \Omega)$ ,  $H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div} \mu; \Omega)$ , etc., equipped with similar norms.

PROPOSITION 2.1 (see [40, 55]). *On Lipschitz domain  $\Omega$ , with  $\varepsilon$  as assumed, for any  $\mathbf{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  or  $\mathbf{v} \in H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div} \varepsilon; \Omega)$ , we have*

$$(2.1) \quad \|\mathbf{curl} \mathbf{v}\|_0^2 + \|\operatorname{div} \varepsilon \mathbf{v}\|_0^2 \geq C \|\mathbf{v}\|_0^2.$$

Put

$$(2.2) \quad \lambda := 1 + \omega^2.$$

The eigenproblem (1.2) is restated as follows: Find  $(\lambda, \mathbf{u} \neq \mathbf{0}) \in \mathbb{R} \times H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  such that

$$(2.3) \quad \mathcal{L}(\mathbf{u}, \mathbf{v}) := (\mu^{-1} \mathbf{curl} \mathbf{u}, \mathbf{v}) + (\operatorname{div} \varepsilon \mathbf{u}, \operatorname{div} \varepsilon \mathbf{v}) + (\varepsilon \mathbf{u}, \mathbf{v}) = \lambda (\varepsilon \mathbf{u}, \mathbf{v}) \\ \forall \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega).$$

The corresponding boundary value problem is as follows:

$$(2.4) \quad \mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u} - \varepsilon \nabla \operatorname{div} \varepsilon \mathbf{u} + \varepsilon \mathbf{u} = \lambda \varepsilon \mathbf{u} \quad \text{in } \Omega, \quad \mathbf{u} \times \mathbf{n} = \mathbf{0}, \quad \operatorname{div} \varepsilon \mathbf{u} = 0 \quad \text{on } \Gamma.$$

Since  $H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  is compactly embedded into  $(L^2(\Omega))^3$  (see [40, 49, 28, 55]), it follows from [29] that the eigenproblem (1.1) has an infinite sequence of eigenvalues  $0 \leq \omega_1^2 \leq \omega_2^2 \leq \dots \nearrow +\infty$ . As a consequence of Proposition 2.1, all eigenvalues  $\omega^2 > C > 0$  of the eigenproblem (1.1). Thus, from (2.2),

$$(2.5) \quad 1 + C < \lambda_1 \leq \lambda_2 \leq \dots \nearrow +\infty.$$

Closely related to the eigenproblem (2.4), the source problem reads as follows: Given  $\mathbf{f} \in (L^2(\Omega))^3$ , find  $\mathbf{z}$  such that

$$(2.6) \quad \mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{z} - \varepsilon \nabla \operatorname{div} \varepsilon \mathbf{z} + \varepsilon \mathbf{z} = \varepsilon \mathbf{f} \quad \text{in } \Omega, \quad \mathbf{z} \times \mathbf{n} = \mathbf{0}, \quad \operatorname{div} \varepsilon \mathbf{z} = 0 \quad \text{on } \Gamma,$$

for which the variational statement is to find  $\mathbf{z} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  such that

$$(2.7) \quad \mathcal{L}(\mathbf{z}, \mathbf{v}) = (\varepsilon \mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega).$$

With (2.7), a linear operator  $\mathbb{T} : (L^2(\Omega))^3 \rightarrow H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  is defined as follows: for any given  $\mathbf{f} \in (L^2(\Omega))^3$ ,  $\mathbf{z} = \mathbb{T} \mathbf{f} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  satisfies

$$(2.8) \quad \mathcal{L}(\mathbb{T} \mathbf{f}, \mathbf{v}) = (\varepsilon \mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega).$$

From Proposition 2.1 it can be easily verified that the linear operator  $\mathbb{T}$  is bounded from  $(L^2(\Omega))^3$  to  $H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ . Moreover,  $\mathbb{T}$  is symmetric positive definite with respect to both the  $\varepsilon$ -weighted  $L^2$  inner product  $(\cdot, \cdot)_\varepsilon$  and the  $\mathcal{L}$ -induced inner product  $(\cdot, \cdot)_\mathcal{L} := \mathcal{L}(\cdot, \cdot)$ . Furthermore,  $\mathbb{T} : (L^2(\Omega))^3 \rightarrow (L^2(\Omega))^3$  is compact (due to the compact embedding of  $H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega) \hookrightarrow (L^2(\Omega))^3$ ). Also,  $\mathbb{T} :$

$H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega) \rightarrow H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  is compact. Clearly, an eigenpair  $(\lambda, \mathbf{u} \neq \mathbf{0})$  of (2) if and only if  $(\nu = 1/\lambda, \mathbf{u} \neq \mathbf{0})$  is an eigenpair of  $\mathbb{T}$ , i.e.,

$$(2.9) \quad \mathbb{T}\mathbf{u} = \nu \mathbf{u} \quad \text{with } \nu = 1/\lambda,$$

and the sequence of eigenvalues of  $\mathbb{T}$  satisfies

$$(2.10) \quad 0 < \dots \leq \nu_2 \leq \nu_1 < 1/(1+C) < 1.$$

**3. The finite element method.** In this section we define the finite element method. Let  $\Omega$  and  $\Omega_j$ ,  $1 \leq j \leq J$ , be simply connected bounded polyhedra in  $\mathbb{R}^3$ , with connected boundaries  $\Gamma = \partial\Omega$  and  $\partial\Omega_j$ . Denote by  $\mathcal{T}_h$  the conforming (also conforming to  $\Gamma$  and  $\partial\Omega_j$ ) triangulation of  $\Omega$  into shape-regular and quasi-uniform tetrahedra [24], where  $h = \max_{K \in \mathcal{T}_h} h_K$  and  $h_K$  is the diameter of  $K$ . Let  $\mathcal{F}_h$  be the collection of all element faces,  $\mathcal{F}_h^\Gamma$  the collection of all the element faces on  $\Gamma$ , and  $\mathcal{F}_h^{\text{Inter}}$  the set of all the element faces on the interfaces of the discontinuous  $\varepsilon$ . Denote by  $\mathcal{F}_h^0 = \mathcal{F}_h \setminus (\mathcal{F}_h^\Gamma \cup \mathcal{F}_h^{\text{Inter}})$  the collection of all interior element faces which are not on the domain boundary and the interfaces. Denote by  $P_l(K)$  the space of polynomials on  $K$  of total degree not greater than  $l$ , where  $l \geq 1$  is an integer. Introduce the element bubble  $b_K \in H_0^1(\Omega)$ . A typical element bubble is  $b_K = \lambda_1 \lambda_2 \lambda_3 \lambda_4$ , where  $\lambda_i$  denotes the  $i$ th basis of  $P_1(K)$  associated with the  $i$ th vertex of  $K$ ,  $1 \leq i \leq 4$ . We shall assume that  $\varepsilon = (\varepsilon_{ij})$ ,  $\varepsilon_{ij}$  is a piecewise polynomial with respect to the partition  $\mathcal{P}$  of  $\Omega$ , and it is also piecewise on  $\mathcal{T}_h$ , i.e.,  $\varepsilon_{ij}|_K$  is a polynomial for  $K \in \mathcal{T}_h$ . For a general piecewise smooth  $\varepsilon$ , we can consider its finite element interpolation of piecewise polynomials as a replacement. On  $K \in \mathcal{T}_h$ , we introduce

$$(3.1) \quad \mathbf{P}_{l-1}(\varepsilon; K) = \operatorname{span}\{\varepsilon(P_{l-1}(K))^3, (P_{l-1}(K))^3\},$$

$$(3.2) \quad \Phi_h = \{v \in (H_0^1(\Omega))^3 : v|_K \in b_K \mathbf{P}_{l-1}(\varepsilon; K) \quad \forall K \in \mathcal{T}_h\},$$

$$(3.3) \quad V_h^l = \{q \in L^2(\Omega) : q|_K \in P_l(K) \quad \forall K \in \mathcal{T}_h\}.$$

We define the  $H^1$ -conforming finite element space  $U_h$  for the solution

$$(3.4) \quad U_h = (V_h^l \cap H^1(\Omega))^3 + \Phi_h, \quad W_h^c = (V_h^l \cap H^1(\Omega))^3.$$

We should emphasize that even if  $\varepsilon$  is discontinuous,  $U_h$  is always continuous and  $H^1$ -conforming over the whole  $\Omega$ . An example for  $U_h$  is  $l = 1$ . In that case,  $U_h$  is an  $H^1$ -conforming linear element, enriched with some element bubbles. The number of element bubbles is up to the degree of the polynomial  $\varepsilon|_K$ . For example, for piecewise constant  $\varepsilon$  and  $l = 1$ , there are  $1 \times 3$  element bubbles for the function  $\mathbf{v} = (v_1, v_2, v_3) \in U_h$  on each element  $K$ . Since all element bubbles are in  $(H_0^1(K))^3$ , they can be element locally eliminated by a standard static condensation procedure before the implementation of the finite element method. The static condensation procedure is a technique to eliminate the element bubbles. Note that any function of  $\Phi_h$  is a function of  $(H_0^1(K))^3$  for all  $K \in \mathcal{T}_h$  and that any function  $\mathbf{v}_h$  of  $U_h$  from (3.4) can be written as the sum of two parts, i.e.,  $\mathbf{v}_h = \mathbf{v}_h^l + \mathbf{v}_h^b$ , where  $\mathbf{v}_h^l \in (V_h^l \cap H^1(\Omega))^3$ ,  $\mathbf{v}_h^b \in \Phi_h$ . We refer to [15, pp. 248–249] for a description of the realization of such a procedure for the MINI element of the Stokes problem. Thus,  $U_h$  is in essence a standard nodal-continuous Lagrange element of degree  $l$ .

Moreover, we introduce two finite element spaces  $Q_h$  and  $W_h$  for the definitions of two  $L^2$  projections  $\check{R}_h$  and  $R_h$ , respectively, as follows:

$$(3.5) \quad Q_h = V_h^l \cap H_0^1(\Omega),$$

$$(3.6) \quad W_h = V_h^l \cap H(\mathbf{curl}; \Omega).$$

Here,  $Q_h$  is  $H^1$ -conforming finite element space, while  $W_h$  is  $H(\mathbf{curl}; \Omega)$ -conforming finite element space of Nédélec elements of the second family. Classical theory for  $Q_h$  and  $W_h$  can be found, respectively, in [24, 19] and [51, 49, 45, 17]. Note that the choices of (3.5) and (3.6) correspond to the natural regularity  $\operatorname{div} \varepsilon \mathbf{u} \in H_0^1(\Omega)$  and  $\mu^{-1} \mathbf{curl} \mathbf{u} \in H(\mathbf{curl}; \Omega)$  for the solution  $\mathbf{u}$ . For any given  $\mathbf{v} \in (L^2(\Omega))^3$ , we define  $\check{R}_h(\operatorname{div} \varepsilon \mathbf{v}) \in Q_h$  and  $R_h(\mu^{-1} \mathbf{curl} \mathbf{v}) \in W_h$  in the following:

$$(3.7) \quad (\check{R}_h(\operatorname{div} \varepsilon \mathbf{v}), q) = -(\mathbf{v}, \varepsilon \nabla q) \quad \forall q \in Q_h,$$

$$(3.8) \quad (R_h(\mu^{-1} \mathbf{curl} \mathbf{v}), \mathbf{w})_\mu = (\mathbf{v}, \mathbf{curl} \mathbf{w}) \quad \forall \mathbf{w} \in W_h.$$

Note that, in general,  $R_h$  and  $\check{R}_h$  are not genuine  $L^2$  projectors, but they are indeed when  $\mathbf{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ . In that case,

$$(3.9) \quad (\check{R}_h(\operatorname{div} \varepsilon \mathbf{v}), q) = (\operatorname{div} \varepsilon \mathbf{v}, q) \quad \forall q \in Q_h,$$

$$(3.10) \quad (R_h(\mu^{-1} \mathbf{curl} \mathbf{v}), \mathbf{w})_\mu = (\mu^{-1} \mathbf{curl} \mathbf{v}, \mathbf{w})_\mu \quad \forall \mathbf{w} \in W_h,$$

and there hold

$$(3.11) \quad \|R_h(\mu^{-1} \mathbf{curl} \mathbf{v})\|_{0,\mu} \leq \|\mu^{-1} \mathbf{curl} \mathbf{v}\|_{0,\mu}, \quad \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{v})\|_0 \leq \|\operatorname{div} \varepsilon \mathbf{v}\|_0.$$

So, we will simply call  $R_h$  and  $\check{R}_h$   $L^2$  projectors.

The  $H^1$ -conforming finite element method we propose is to find  $(\lambda_h, \mathbf{u}_h \neq \mathbf{0}) \in \mathbb{R} \times U_h$  such that

$$(3.12) \quad \mathcal{L}_h(\mathbf{u}_h, \mathbf{v}_h) = \lambda_h(\varepsilon \mathbf{u}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in U_h,$$

where

$$(3.13) \quad \begin{aligned} \mathcal{L}_h(\mathbf{u}_h, \mathbf{v}_h) = & (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}_h), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_h))_\mu \\ & + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}_h), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_h)) + (\varepsilon \mathbf{u}_h, \mathbf{v}_h). \end{aligned}$$

The method is nonconforming, not only due to the nonconformity of  $U_h$ , but also due to the introduction of the  $L^2$  projections in the bilinear form. Note that the bilinear form  $\mathcal{L}_h(\mathbf{u}, \mathbf{v})$  is well-defined for all  $\mathbf{u}, \mathbf{v} \in (L^2(\Omega))^3$ . As such, even when the exact solution is only  $L^2$ , a correct convergence would be expected.

Let  $\mathbb{T}_h : (L^2(\Omega))^3 \rightarrow U_h$  be a bounded linear operator, defined in the following way: Given  $\mathbf{f} \in (L^2(\Omega))^3$ , find  $\mathbb{T}_h \mathbf{f} \in U_h$  such that

$$(3.14) \quad \mathcal{L}_h(\mathbb{T}_h \mathbf{f}, \mathbf{v}_h) = (\varepsilon \mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in U_h.$$

As in the continuous case, analogously, an eigenpair  $(\lambda_h, \mathbf{u}_h)$  of (3.12) if and only if  $(\nu_h = 1/\lambda_h, \mathbf{u}_h)$  is an eigenpair of  $\mathbb{T}_h$ , i.e.,

$$(3.15) \quad \mathbb{T}_h \mathbf{u}_h = \nu_h \mathbf{u}_h \quad \text{with } \nu_h = 1/\lambda_h.$$

Since the most interesting solution is the  $H^r$  space solution for some  $r \leq 1$ , we shall only focus on the analysis for the linear element method, i.e.,  $l = 1$ . However, all the analysis will be valid for higher-order element methods, with very few modifications.



**4. The Fortin-type interpolation and the Inf-Sup inequality.** In this section we shall construct two Fortin-type interpolations and an Inf-Sup inequality associated with the following trilinear form, which will be the main tools for analyzing the stability and the error bounds. We introduce a trilinear form over  $(L^2(\Omega))^3 \times H_0^1(\Omega) \times H(\mathbf{curl}; \Omega)$  as follows:

$$(4.1) \quad b(\mathbf{v}; (p, \mathbf{w})) = (\mathbf{v}, \mathbf{curl} \mathbf{w}) - (\varepsilon \mathbf{v}, \nabla p) : (L^2(\Omega))^3 \times H_0^1(\Omega) \times H(\mathbf{curl}; \Omega) \rightarrow \mathbb{R}.$$

This trilinear form relates to the curlcurl-graddiv operator in (1.1) by inserting  $\mathbf{w} = \mu^{-1} \mathbf{curl} \mathbf{u}$  and  $p = \operatorname{div} \varepsilon \mathbf{u}$ . The role of  $b(\mathbf{v}; (p, \mathbf{w}))$  is for dealing with the nonconformity caused by the  $L^2$  projections.

**4.1. Properties of the kernel of  $b$ .** We shall investigate the properties of the kernel set of the trilinear form  $b$ , so that we can arrive at an Inf-Sup condition and a Fortin-type interpolation in the  $L^2$  orthogonal complement of the kernel of  $b$  in  $U_h$ . Some of the properties established for the kernel of  $b$  will also be used in the error estimates.

Let the kernel set of  $b$  be defined by

$$(4.2) \quad \mathcal{K}_h(b) = \{\mathbf{v} \in U_h : b(\mathbf{v}; (p, \mathbf{w})) = 0 \quad \forall p \in Q_h, \quad \forall \mathbf{w} \in W_h\}.$$

For the finite dimensional space  $U_h$  we can have the following orthogonal decomposition with respect to the  $\varepsilon$ -weighted  $L^2$  inner product  $(\cdot, \cdot)_\varepsilon$ :

$$(4.3) \quad U_h = \mathcal{K}_h(b) + \mathcal{K}_h(b)^\perp,$$

satisfying

$$(4.4) \quad (\varepsilon \mathbf{u}, \mathbf{v}) = 0 \quad \forall \mathbf{u} \in \mathcal{K}_h(b), \forall \mathbf{v} \in \mathcal{K}_h(b)^\perp.$$

Since, in terms of  $R_h$  and  $\check{R}_h$ ,

$$(4.5) \quad b(\mathbf{v}; (p, \mathbf{w})) = (R_h(\mu^{-1} \mathbf{curl} \mathbf{v}), \mathbf{w})_\mu + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{v}), q) \quad \forall q \in Q_h, \forall \mathbf{w} \in W_h,$$

the kernel set  $\mathcal{K}_h(b)$  of  $b$  can also be equivalently defined as follows:

$$(4.6) \quad \mathcal{K}_h(b) = \{\mathbf{v} \in U_h : R_h(\mu^{-1} \mathbf{curl} \mathbf{v}) = \mathbf{0}, \quad \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}) = 0\}.$$

**PROPOSITION 4.1** (see [40, 2]). *We have the following  $\varepsilon$ -weighted  $L^2$  orthogonal decomposition and the  $\mu$ -weighted  $L^2$  orthogonal decomposition:*

$$(4.7) \quad \begin{aligned} (L^2(\Omega))^3 &= \varepsilon^{-1} \mathbf{curl}(H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega)) + \nabla H_0^1(\Omega) \\ &= H_0(\operatorname{div}^0 \mu; \Omega) + \nabla(H^1(\Omega)/\mathbb{R}). \end{aligned}$$

**Assumption 1.** We require that the following two continuous embeddings hold for some  $1 \geq r \geq 0$ :

$$(4.8) \quad H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega), H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div} \mu; \Omega) \hookrightarrow \prod_{j=1}^J (H^r(\Omega_j))^3,$$

where for any  $\mathbf{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ , or  $\mathbf{v} \in H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div} \mu; \Omega)$  we have

$$(4.9) \quad \sum_{j=1}^J \|\mathbf{v}\|_{r, \Omega_j} \leq C \|\mathbf{v}\|_{\mathbf{curl}; \operatorname{div} \varepsilon} \quad \text{or} \quad C \|\mathbf{v}\|_{\mathbf{curl}; \operatorname{div} \mu}.$$

Above,  $r$  stands for the piecewise regularity exponent.

In the case where  $\varepsilon$  and  $\mu$  belong to  $(W^{1,\infty}(\Omega))^{3 \times 3}$ , the above assumption is known to hold in  $(H^r(\Omega))^3$ . In fact,  $r > 1/2$  for Lipschitz polyhedra [2] and  $r = 1/2$  for general Lipschitz domains [28]. On the other hand, for non-Lipschitz domains,  $r$  may be less than  $1/2$ ; see [30]. Throughout this paper we do not require  $r > 1/2$ , although we have assumed a Lipschitz polyhedron  $\Omega$ . In addition, the above two continuous embedding may have different regularity  $r$ . In fact, (4.8) may come from the regularity of the second-order elliptic interface problem of Laplacian, corresponding to Dirichlet and Neumann boundary conditions [2, 43]. As pointed out in [34], for interface problems, not only is the global regularity commonly very low [16], but also the piecewise regularity is possibly very low.

To establish the Fortin interpolation stated in Theorem 4.2 later on using  $H^1$ -conforming and nodal-continuous elements, with Assumption 1 at hand, we need to introduce a real number to indicate the global regularity

$$r_0 = \min(r, 1/2 - \epsilon),$$

where the parameter  $\epsilon$  can be any given small positive constant in the interval  $(0, 1/2)$ . All the analysis and theoretical results throughout this paper hold for  $r_0$ . Below we make some remarks on the relationship between the global regularity  $r_0$  and the piecewise regularity  $r$ . If  $r < 1/2$ , noting the fact that  $H^r(\Omega_j) = H_0^r(\Omega_j)$ , we have  $\prod_{j=1}^J (H^r(\Omega_j))^3 = (H^r(\Omega))^3$ , and we take  $r_0 = r$ . If  $1 \geq r \geq 1/2$ , noting the fact that  $H^r(\Omega_j)$  continuously embeds into  $H^{1/2-\epsilon}(\Omega_j)$  for any positive constant  $\epsilon > 0$ , we have that  $\prod_{j=1}^J (H^r(\Omega_j))^3$  continuously embeds into  $(H^{1/2-\epsilon}(\Omega))^3$ , and we take  $r_0 = 1/2 - \epsilon$ . In other words, from the continuous embeddings in Assumption 1, we find that the following hold in terms of the global regularity  $r_0$  as defined:

$$(4.8)' \quad H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega), H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div} \mu; \Omega) \hookrightarrow \prod_{j=1}^J (H^r(\Omega_j))^3 \hookrightarrow (H^{r_0}(\Omega))^3,$$

where for any  $\mathbf{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ , or  $\mathbf{v} \in H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div} \mu; \Omega)$  we have

$$(4.9)' \quad \|\mathbf{v}\|_{r_0} \leq C \sum_{j=1}^J \|\mathbf{v}\|_{r, \Omega_j} \leq C \|\mathbf{v}\|_{\mathbf{curl}; \operatorname{div} \varepsilon} \quad \text{or} \quad C \|\mathbf{v}\|_{\mathbf{curl}; \operatorname{div} \mu}.$$

We should note that if  $r$  itself stands for the global regularity (e.g., this often happens for  $\varepsilon = \mu = 1$ ), we just take  $r_0 = r$ , without the restriction being less than  $1/2$  on  $r_0$ . In addition, under some situations where the piecewise regularity  $r > 1$ , we may also take  $r_0 = r$  itself. See further remarks in Remark 4.1 after Theorem 4.2.

LEMMA 4.1. *For any given  $\mathbf{f}$ , if  $\mathbf{f} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ , then, under Assumption 1,*

$$(4.10) \quad \mathbf{f} = \nabla A + \varepsilon^{-1} \mathbf{curl} \mathbf{B},$$

where  $A \in \prod_{j=1}^J H^{1+r}(\Omega_j) \cap H_0^1(\Omega)$  and  $\mathbf{B}, \mathbf{curl} \mathbf{B} \in \prod_{j=1}^J (H^r(\Omega_j))^3$ , satisfy

$$(4.11) \quad \sum_{j=1}^J \|A\|_{1+r, \Omega_j} + \|\mathbf{B}\|_{r, \Omega_j} + \|\mathbf{curl} \mathbf{B}\|_{r, \Omega_j} \leq C \|\mathbf{f}\|_{\mathbf{curl}; \operatorname{div} \varepsilon}.$$

*Proof.* From Proposition 4.1 we can write  $\mathbf{f}$  into the  $\varepsilon$ -weighted  $L^2$  orthogonal decomposition  $\mathbf{f} = \nabla A + \varepsilon^{-1} \mathbf{curl} \mathbf{B}$ , where  $A \in H_0^1(\Omega)$ , and  $\mathbf{B} \in H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega)$ , and  $A$  satisfies  $\operatorname{div} \varepsilon \nabla A = \operatorname{div} \varepsilon \mathbf{f}$  in  $\Omega$  and  $A = 0$  on  $\Gamma$ . Since  $\nabla A \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ , from Assumption 1 we know that  $A \in \prod_{j=1}^J H^{1+r}(\Omega_j)$  satisfies  $\sum_{j=1}^J \|A\|_{1+r, \Omega_j} \leq C \|\mathbf{f}\|_{0, \operatorname{div} \varepsilon}$ . Note that from Assumption 1,  $\mathbf{f} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega) \hookrightarrow \prod_{j=1}^J (H^r(\Omega_j))^3$ , satisfying  $\sum_{j=1}^J \|\mathbf{f}\|_{r, \Omega_j} \leq C \|\mathbf{f}\|_{\mathbf{curl}, \operatorname{div} \varepsilon}$ . Also, note that from Proposition 2.1,  $\|\mathbf{f}\|_0 \leq C \|\mathbf{f}\|_{\mathbf{curl}, \operatorname{div} \varepsilon}$ . Since  $\varepsilon^{-1} \mathbf{curl} \mathbf{B} = \mathbf{f} - \nabla A \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ , from Assumption 1 we have  $\varepsilon^{-1} \mathbf{curl} \mathbf{B} \in \prod_{j=1}^J (H^r(\Omega_j))^3$ , satisfying  $\sum_{j=1}^J \|\varepsilon^{-1} \mathbf{curl} \mathbf{B}\|_{r, \Omega_j} \leq C \|\mathbf{f}\|_{\mathbf{curl}, \operatorname{div} \varepsilon}$  and  $\|\mathbf{curl} \mathbf{B}\|_0 \leq C \|\mathbf{f}\|_0$ . Since  $\varepsilon|_{\Omega_j} \in (W^{1, \infty}(\Omega_j))^{3 \times 3}$ , we further have  $\mathbf{curl} \mathbf{B} \in \prod_{j=1}^J (H^r(\Omega_j))^3$ , satisfying  $\sum_{j=1}^J \|\mathbf{curl} \mathbf{B}\|_{r, \Omega_j} \leq C \|\mathbf{f}\|_{\mathbf{curl}, \operatorname{div} \varepsilon}$ . While, Assumption 1 also gives  $\mathbf{B} \in \prod_{j=1}^J (H^r(\Omega_j))^3$ , satisfying  $\sum_{j=1}^J \|\mathbf{B}\|_{r, \Omega_j} \leq C \|\mathbf{B}\|_{\mathbf{curl}, \operatorname{div} \mu} = C \|\mathbf{curl} \mathbf{B}\|_0 \leq C \|\mathbf{f}\|_0 \leq C \|\mathbf{f}\|_{\mathbf{curl}, \operatorname{div} \varepsilon}$ . We then obtain (4.11).  $\square$

**THEOREM 4.1.** Assume Assumption 1 and  $\mathbf{f} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ . We have

$$(4.12) \quad |(\varepsilon \mathbf{f}, \mathbf{v}_{0,h})| \leq Ch^r \|\mathbf{f}\|_{\mathbf{curl}, \operatorname{div} \varepsilon} \|\mathbf{v}_{0,h}\|_0 \quad \forall \mathbf{v}_{0,h} \in \mathcal{K}_h(b).$$

*Proof.* From Lemma 4.1, we write  $\mathbf{f}$  as  $\mathbf{f} = \varepsilon^{-1} \mathbf{curl} \mathbf{B} - \nabla A$ , where  $A \in \prod_{j=1}^J H^{1+r}(\Omega_j) \cap H_0^1(\Omega)$  and  $\mathbf{B}, \mathbf{curl} \mathbf{B} \in \prod_{j=1}^J (H^r(\Omega_j))^3$  satisfy  $\sum_{j=1}^J \|A\|_{1+r, \Omega_j} + \|\mathbf{B}\|_{r, \Omega_j} + \|\mathbf{curl} \mathbf{B}\|_{r, \Omega_j} \leq C \|\mathbf{f}\|_{\mathbf{curl}, \operatorname{div} \varepsilon}$ . Since  $\mathbf{v}_{0,h}$  satisfies  $b(\mathbf{v}_{0,h}; (I_h A, \mathbf{J}_h \mathbf{B})) = 0$ , where  $I_h A \in Q_h$  and  $\mathbf{J}_h \mathbf{B} \in W_h$ , respectively, stand for the finite element interpolations of  $A$  and  $\mathbf{B}$ , we have

$$(4.13) \quad (\varepsilon \mathbf{f}, \mathbf{v}_{0,h}) = (\mathbf{curl} \mathbf{B} - \varepsilon \nabla A, \mathbf{v}_{0,h}) = b(\mathbf{v}_{0,h}; (A, \mathbf{B})) = b(\mathbf{v}_{0,h}; (A - I_h A, \mathbf{B} - \mathbf{J}_h \mathbf{B})),$$

where

$$(4.14) \quad (\mathbf{curl}(\mathbf{B} - \mathbf{J}_h \mathbf{B}), \mathbf{v}_{0,h}) \leq Ch^r \sum_{j=1}^J \|\mathbf{curl} \mathbf{B}\|_{r, \Omega_j} \|\mathbf{v}_{0,h}\|_0,$$

$$(4.15) \quad -(\varepsilon \nabla(A - I_h A), \mathbf{v}_{0,h}) \leq Ch^r \sum_{j=1}^J \|A\|_{1+r, \Omega_j} \|\mathbf{v}_{0,h}\|_0.$$

It follows that

$$(4.16) \quad |(\varepsilon \mathbf{f}, \mathbf{v}_{0,h})| \leq Ch^r \left( \sum_{j=1}^J \|\mathbf{curl} \mathbf{B}\|_{r, \Omega_j} + \|A\|_{1+r, \Omega_j} \right) \|\mathbf{v}_{0,h}\|_0. \quad \square$$

**4.2. The Fortin-type interpolation and the Inf-Sup inequality.** We first introduce a mesh-dependent norm for the establishment of the Inf-Sup inequality. For any  $\mathbf{v}_h \in U_h$ , with  $\mathbf{v}_h = \mathbf{v}_{0,h} + \mathbf{v}_{0,h}^\perp$ ,  $\mathbf{v}_{0,h} \in \mathcal{K}_h(b)$ , and  $\mathbf{v}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp$ , we define

$$(4.17) \quad \begin{aligned} \|\mathbf{v}_h\|_h^2 &= h^{-2r} \|\mathbf{v}_{0,h}\|_0^2 + \|\mathbf{v}_{0,h}^\perp\|_0^2 + \sum_{K \in \mathcal{T}_h} h_K^{2-2r} \|\operatorname{div} \varepsilon \mathbf{v}_h\|_{0,K}^2 \\ &\quad + \sum_{K \in \mathcal{T}_h} h_K^{2-2r} \|\mathbf{curl} \mathbf{v}_h\|_{0,K}^2 \\ &\quad + \sum_{F \in \mathcal{F}_h^\Gamma} h_F^{1-2r} \int_F |\mathbf{v}_h \times \mathbf{n}|^2 + \sum_{F \in \mathcal{F}_h^{\operatorname{Inter}}} h_F^{1-2r} \int_F |[\varepsilon \mathbf{v}_h \cdot \mathbf{n}]|^2, \end{aligned}$$

where  $h_F$  is the diameter of the face  $F$ ,  $0 \leq r \leq 1$  comes from Assumption 1, and  $[q]$  denotes the jump.

LEMMA 4.2. *For all  $\mathbf{v}_h \in U_h$ , the following holds:*

$$(4.18) \quad \|\mathbf{v}_h\|_h \geq C \|\mathbf{v}_{0,h}^\perp\|_h.$$

*Proof.* In fact, since  $\mathbf{v}_h = \mathbf{v}_{0,h} + \mathbf{v}_{0,h}^\perp$ , with  $\mathbf{v}_{0,h} \in \mathcal{X}_h(b)$  and  $\mathbf{v}_{0,h}^\perp \in \mathcal{X}_h(b)^\perp$ , from the element-local inverse estimates for the functions of finite dimensional spaces, i.e.,

$$(4.19) \quad \|v\|_{1,K} \leq Ch_K^{-1} \|v\|_{0,K},$$

we have

$$\begin{aligned} h_K^{2-2r} \|\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp\|_{0,K}^2 &\leq 2h_K^{2-2r} \|\operatorname{div} \varepsilon \mathbf{v}_h\|_{0,K}^2 + 2h_K^{2-2r} \|\operatorname{div} \varepsilon \mathbf{v}_{0,h}\|_{0,K}^2, \\ h_K^{2-2r} \|\operatorname{div} \varepsilon \mathbf{v}_{0,h}\|_{0,K}^2 &\leq Ch_K^{2-2r} \|\mathbf{v}_{0,h}\|_{1,K}^2 \leq Ch_K^{-2r} \|\mathbf{v}_{0,h}\|_{0,K}^2, \\ h_K^{2-2r} \|\operatorname{curl} \mathbf{v}_{0,h}^\perp\|_{0,K}^2 &\leq 2h_K^{2-2r} \|\operatorname{curl} \mathbf{v}_h\|_{0,K}^2 + 2h_K^{2-2r} \|\operatorname{curl} \mathbf{v}_{0,h}\|_{0,K}^2, \\ h_K^{2-2r} \|\operatorname{curl} \mathbf{v}_{0,h}\|_{0,K}^2 &\leq Ch_K^{2-2r} \|\mathbf{v}_{0,h}\|_{1,K}^2 \leq Ch_K^{-2r} \|\mathbf{v}_{0,h}\|_{0,K}^2, \\ h_F^{1-2r} \int_F |\mathbf{v}_{0,h}^\perp \times \mathbf{n}|^2 &\leq 2h_F^{1-2r} \int_F |\mathbf{v}_h \times \mathbf{n}|^2 + 2h_F^{1-2r} \int_F |\mathbf{v}_{0,h} \times \mathbf{n}|^2, \\ h_F^{1-2r} \int_F |\mathbf{v}_{0,h} \times \mathbf{n}|^2 &\leq Ch_K^{-2r} \|\mathbf{v}_{0,h}\|_{0,K}^2, \\ h_F^{1-2r} \int_F |[\varepsilon \mathbf{v}_{0,h}^\perp \cdot \mathbf{n}]|^2 &\leq 2h_F^{1-2r} \int_F |[\varepsilon \mathbf{v}_h \cdot \mathbf{n}]|^2 + 2h_F^{1-2r} \int_F |[\varepsilon \mathbf{v}_{0,h} \cdot \mathbf{n}]|^2, \\ h_F^{1-2r} \int_F |[\varepsilon \mathbf{v}_{0,h} \cdot \mathbf{n}]|^2 &\leq Ch_K^{-2r} \|\mathbf{v}_{0,h}\|_{0,K}^2 + h_{K'}^{-2r} \|\mathbf{v}_{0,h}\|_{0,K'}^2, \quad \square \end{aligned}$$

where  $F = \partial K \cap \partial K'$ , with  $K, K' \in \mathcal{T}_h$ . Combining all these, we obtain the result (4.18).

By the triangle inequality, moreover, we obtain from Lemma 4.2  $\|\mathbf{v}_{0,h}\|_h \leq \|\mathbf{v}_h\|_h + \|\mathbf{v}_{0,h}^\perp\|_h \leq C \|\mathbf{v}_h\|_h$ . On the other hand, by the triangle inequality again, we have  $\|\mathbf{v}_h\|_h \leq \|\mathbf{v}_{0,h}\|_h + \|\mathbf{v}_{0,h}^\perp\|_h$ . So,  $\|\mathbf{v}_{0,h}\|_h + \|\mathbf{v}_{0,h}^\perp\|_h$  is equivalent to  $\|\mathbf{v}_h\|_h$  over  $U_h$ . Thus, there exists some orthogonality between  $\mathbf{v}_{0,h}$  and  $\mathbf{v}_{0,h}^\perp$ . Here we should note that the mesh-dependent norm  $\|\cdot\|_h$  is introduced only for theoretical analysis and never involved with the implementation of the proposed finite element method.

PROPOSITION 4.2. *We have the following inclusions:*

$$(4.20) \quad \operatorname{curl} W_h|_K, \varepsilon \nabla Q_h|_K \subset \mathbf{P}_{l-1}(\varepsilon; K) \quad \forall K \in \mathcal{T}_h.$$

*Proof.* By the definitions (3.1), (3.3), (3.5), and (3.6), we conclude.  $\square$

THEOREM 4.2. *Under Assumption 1, for any given  $\mathbf{v} \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ , there is a Fortin-type interpolation  $\tilde{\mathbf{v}} \in U_h \cap H_0(\operatorname{curl}; \Omega)$  of  $\mathbf{v}$  such that*

$$(4.21) \quad b(\tilde{\mathbf{v}}; (p, \mathbf{w})) = b(\mathbf{v}; (p, \mathbf{w})) \quad \forall p \in Q_h, \forall \mathbf{w} \in W_h,$$

$$(4.22) \quad \|\tilde{\mathbf{v}}\|_h \leq C \|\mathbf{v}\|_{\operatorname{curl}; \operatorname{div} \varepsilon},$$

where  $\|\cdot\|_h$  is defined by (4.17) but with  $r$  replaced by  $r_0$ , and the following interpolation properties (4.23)–(4.25) hold in terms of the global regularity  $r_0$  as defined earlier:

$$(4.23) \quad \int_K (\tilde{\mathbf{v}} - \mathbf{v}) \cdot \mathbf{w} = 0 \quad \forall \mathbf{w} \in \mathbf{P}_{l-1}(\varepsilon; K), \forall K \in \mathcal{T}_h,$$

$$(4.24) \quad \|\mathbf{v} - \tilde{\mathbf{v}}\|_0 \leq Ch^{r_0} \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon},$$

$$(4.25) \quad \left( \sum_{K \in \mathcal{T}_h} h_K^{-2r_0} \|\mathbf{v} - \tilde{\mathbf{v}}\|_{0,K}^2 \right)^{1/2} + \|\tilde{\mathbf{v}}\|_{r_0} \leq C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon}.$$

*Proof.* First, from Assumption 1 (see (4.8) and (4.9)),

$$\mathbf{v} \in \prod_{j=1}^J (H^r(\Omega_j))^3 \hookrightarrow (H^{r_0}(\Omega))^3,$$

satisfying

$$\|\mathbf{v}\|_{r_0} \leq C \sum_{j=1}^J \|\mathbf{v}\|_{r, \Omega_j} \leq C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon}.$$

From the finite element interpolation theory [27, 12, 41, 54],<sup>1</sup> we can construct  $\mathbf{v}_h \in (V_h^l \cap H^1(\Omega))^3 \cap H_0(\mathbf{curl}; \Omega) \subset U_h$ , the finite element interpolation of  $\mathbf{v} \in (H^{r_0}(\Omega))^3$ , satisfying

$$(4.26) \quad \|\mathbf{v} - \mathbf{v}_h\|_0 \leq Ch^{r_0} \|\mathbf{v}\|_{r_0} \leq Ch^{r_0} \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon},$$

$$(4.27) \quad \left( \sum_{K \in \mathcal{T}_h} h_K^{-2r_0} \|\mathbf{v} - \mathbf{v}_h\|_{0,K}^2 \right)^{1/2} + \|\mathbf{v}_h\|_{r_0} \leq C \|\mathbf{v}\|_{r_0} \leq C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon}.$$

We then define  $\tilde{\mathbf{v}} \in U_h \cap H_0(\mathbf{curl}; \Omega)$  as follows:

$$(4.28) \quad \tilde{\mathbf{v}}(a) = \mathbf{v}_h(a) \quad \forall \text{ nodes } a \text{ of } P_l(K), \forall K \in \mathcal{T}_h,$$

$$(4.29) \quad \int_K (\tilde{\mathbf{v}} - \mathbf{v}) \cdot \mathbf{w} = 0 \quad \forall \mathbf{w} \in \mathbf{P}_{l-1}(\varepsilon; K), \forall K \in \mathcal{T}_h.$$

It is not difficult to verify that  $\tilde{\mathbf{v}} \in U_h \cap H_0(\mathbf{curl}; \Omega)$  is uniquely determined by (4.28) and (4.29). In fact, writing  $\tilde{\mathbf{v}} = \mathbf{v}^L + \mathbf{v}^B$ , with the part  $\mathbf{v}^L \in (V_h^l \cap H^1(\Omega))^3 \cap H_0(\mathbf{curl}; \Omega)$  and with the bubble part  $\mathbf{v}^B \in \Phi_h$ , we see that the part  $\mathbf{v}^L$  is determined by (4.28) and the bubble part  $\mathbf{v}^B$  by (4.29) element by element. In other words,

$$(4.30) \quad \mathbf{v}^L = \mathbf{v}_h,$$

$$(4.31) \quad \int_K \mathbf{v}^B \cdot \mathbf{w} = \int_K (\mathbf{v} - \mathbf{v}_h) \cdot \mathbf{w} \quad \forall \mathbf{w} \in \mathbf{P}_{l-1}(\varepsilon; K), \forall K \in \mathcal{T}_h.$$

We therefore first conclude that (4.21) holds, since, from (4.29) and Proposition 4.2, we have

$$b(\tilde{\mathbf{v}} - \mathbf{v}; (p, \mathbf{w})) = (\tilde{\mathbf{v}} - \mathbf{v}, \mathbf{curl} \mathbf{w} - \varepsilon \nabla p) = 0 \quad \forall p \in Q_h, \mathbf{w} \in W_h.$$

<sup>1</sup>The vector interpolation with vanishing tangential components can follow from the Scott–Zhang interpolation, since this interpolation preserves conforming boundary values and since the normal vector  $\mathbf{n}$  is constant to each polygonal face of  $\Gamma$ .

On  $K \in \mathcal{T}_h$ , by the standard scaling argument [24, 19, 41], we can have

$$(4.32) \quad \|\mathbf{v}^B\|_{0,K} \leq C \|\mathbf{v} - \mathbf{v}_h\|_{0,K},$$

and we obtain

$$(4.33) \quad \|\tilde{\mathbf{v}} - \mathbf{v}\|_{0,K} \leq C \|\mathbf{v} - \mathbf{v}_h\|_{0,K}.$$

Hence,  $\tilde{\mathbf{v}}$  satisfies the same interpolation error properties (4.26) and (4.27), that is, (4.24) and (4.25) hold. Note that the boundedness of  $\|\tilde{\mathbf{v}}\|_{r_0}$  can be obtained as follows:

$$\begin{aligned} \|\tilde{\mathbf{v}}\|_{r_0} &\leq \|\tilde{\mathbf{v}} - \mathbf{v}_h\|_{r_0} + \|\mathbf{v}_h\|_{r_0} \leq Ch^{-r_0} \|\tilde{\mathbf{v}} - \mathbf{v}_h\|_0 + C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon} \\ &\leq Ch^{-r_0} \|\tilde{\mathbf{v}} - \mathbf{v}\|_0 + Ch^{-r_0} \|\mathbf{v}_h - \mathbf{v}\|_0 + C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon} \leq C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon}. \end{aligned}$$

Moreover, (4.23) comes from the definition (4.29). By the element-local inverse estimates over finite dimensional spaces,  $h_K^{1-t} \|\mathbf{u}\|_{1,K} \leq C \|\mathbf{u}\|_{t,K}$ , for all  $t \in [0, 1]$ , for all  $\mathbf{u} \in U_h$ , and for all  $K \in \mathcal{T}_h$ , we obtain from (4.25) that

$$\sum_{K \in \mathcal{T}_h} h_K^{2-2r_0} \|\text{div} \varepsilon \tilde{\mathbf{v}}\|_{0,K}^2 + \sum_{K \in \mathcal{T}_h} h_K^{2-2r_0} \|\mathbf{curl} \tilde{\mathbf{v}}\|_{0,K}^2 \leq C \|\tilde{\mathbf{v}}\|_{r_0}^2 \leq C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon}^2.$$

Similarly, since  $\mathbf{v} \in H(\text{div} \varepsilon; \Omega)$ , we have

$$\sum_{F \in \mathcal{F}_h^{\text{Inter}}} h_F^{1-2r_0} \int_F |\varepsilon \tilde{\mathbf{v}} \cdot \mathbf{n}|^2 = \sum_{F \in \mathcal{F}_h^{\text{Inter}}} h_F^{1-2r_0} \int_F |\varepsilon (\tilde{\mathbf{v}} - \mathbf{v}) \cdot \mathbf{n}|^2 \leq C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon}^2.$$

On the other hand, since  $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_0^\perp$ , satisfying  $\|\tilde{\mathbf{v}}\|_{0,\varepsilon}^2 = \|\tilde{\mathbf{v}}_0\|_{0,\varepsilon}^2 + \|\tilde{\mathbf{v}}_0^\perp\|_{0,\varepsilon}^2$ , where  $\tilde{\mathbf{v}}_0 \in \mathcal{X}_h(b)$  and  $\tilde{\mathbf{v}}_0^\perp \in \mathcal{X}_h(b)^\perp$ , we have

$$\|\tilde{\mathbf{v}}_0\|_{0,\varepsilon}^2 = (\tilde{\mathbf{v}}_0, \tilde{\mathbf{v}}_0)_\varepsilon = (\tilde{\mathbf{v}}_0, \tilde{\mathbf{v}})_\varepsilon = (\tilde{\mathbf{v}}_0, \tilde{\mathbf{v}} - \mathbf{v})_\varepsilon + (\tilde{\mathbf{v}}_0, \mathbf{v})_\varepsilon.$$

From (4.24) and Theorem 4.1,

$$(\tilde{\mathbf{v}}_0, \tilde{\mathbf{v}} - \mathbf{v})_\varepsilon \leq \|\tilde{\mathbf{v}}_0\|_{0,\varepsilon} \|\tilde{\mathbf{v}} - \mathbf{v}\|_{0,\varepsilon} \leq Ch^{r_0} \|\tilde{\mathbf{v}}_0\|_0 \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon},$$

$$|(\tilde{\mathbf{v}}_0, \mathbf{v})_\varepsilon| \leq Ch^{r_0} \|\tilde{\mathbf{v}}_0\|_0 \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon}.$$

We thus have

$$(4.34) \quad h^{-2r_0} \|\tilde{\mathbf{v}}_0\|_0^2 \leq C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon}^2,$$

$$(4.35) \quad \|\tilde{\mathbf{v}}_0^\perp\|_0^2 \leq 2(\|\tilde{\mathbf{v}}\|_0^2 + \|\tilde{\mathbf{v}}_0\|_0^2) \leq C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon}^2.$$

Hence, combining all the above, we obtain (4.22).  $\square$

We call such  $\tilde{\mathbf{v}}$  the Fortin-type interpolation which satisfies (4.21), mimicking the classical Fortin interpolation for mixed methods [20]. In terms of  $L^2$  projectors  $R_h$  and  $\check{R}_h$ , (4.21) equivalently states

$$(4.36) \quad R_h(\mu^{-1} \mathbf{curl}(\mathbf{v} - \tilde{\mathbf{v}})) = \mathbf{0}, \quad \check{R}_h(\text{div} \varepsilon(\mathbf{v} - \tilde{\mathbf{v}})) = 0.$$

*Remark 4.1.* There are some remarks on the rate  $r_0$  in (4.26) (or that in (4.24)). If  $\mathbf{u} \in (H^r(\Omega))^3$ , the rate  $r_0$  in  $L^2$ -norm can be  $r$ , which may take any value in the interval  $[0, 2]$  (considering the linear element). If  $\mathbf{u} \in (H^1(\Omega))^3 \cap \prod_{j=1}^J (H^r(\Omega_j))^3$  and if  $r \geq 1$ , the rate  $r_0$  in  $L^2$ -norm can still be  $r$  (see Lemma 6.3 in [38]), and similarly, for  $\mathbf{u} \in H(\mathbf{curl}; \Omega) \cap H(\text{div}; \Omega) \cap \prod_{j=1}^J (H^r(\Omega_j))^3$ , where  $H(\mathbf{curl}; \Omega) \cap H(\text{div}; \Omega)$  replaces  $(H^1(\Omega))^3$ , the rate  $r_0$  can be  $r$ , as well, which may take any value in the interval  $[0, 2]$ . Note that the  $\mathbf{u}$  which belongs to  $H(\mathbf{curl}; \Omega) \cap H(\text{div}; \Omega)$  does not necessarily belong to  $H(\mathbf{curl}; \Omega) \cap H(\text{div}; \Omega)$ . If  $\mathbf{u} \in \prod_{j=1}^J (H^r(\Omega_j))^3$  and is discontinuous with either the jump  $[\mathbf{n} \times \mathbf{u}] \neq \mathbf{0}$  or the jump  $[\mathbf{u} \cdot \mathbf{n}] \neq 0$  (albeit  $[(\varepsilon \mathbf{u}) \cdot \mathbf{n}] = 0$ ) or the jump  $[\mathbf{u}] \neq \mathbf{0}$  on some interfacial boundaries, there are two cases: (C1) if  $r < 1/2$ , then the rate  $r_0$  in  $L^2$ -norm is still  $r$ , since  $\prod_{j=1}^J H^r(\Omega_j) = \prod_{j=1}^J H_0^r(\Omega_j) = H_0^r(\Omega) = H^r(\Omega)$ ; (C2) if  $r \geq 1/2$ , the rate  $r_0$  could not be  $r$ , because it seems not to be possible to construct a nodal-continuous interpolation with the rate  $r$  for a discontinuous function which has the piecewise regularity  $r$ . Of course, the rate  $r_0$  can be  $1/2 - \epsilon$  for any small  $\epsilon > 0$ , as remarked after Assumption 1; see (4.8) and (4.9).

PROPOSITION 4.3 (see [40]). *For any given  $\mathbf{w} \in H_0(\text{div}^0 \mu; \Omega)$  and  $p \in L^2(\Omega)$ , there exists a unique solution  $\mathbf{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\text{div}; \Omega)$  to the following problem:*

$$(4.37) \quad \mu^{-1} \mathbf{curl} \mathbf{v} = \mathbf{w}, \quad \text{div} \varepsilon \mathbf{v} = p \quad \text{in } \Omega, \quad \mathbf{v} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

Moreover,  $\mathbf{v} \in \prod_{j=1}^J (H^r(\Omega_j))^3$  from Assumption 1, satisfying

$$(4.38) \quad \sum_{j=1}^J \|\mathbf{v}\|_{r, \Omega_j} \leq C (\|\mathbf{w}\|_0 + \|p\|_0).$$

Now we are able to establish an Inf-Sup inequality in the following. This will mainly rely on the classical orthogonal decomposition of the  $H(\mathbf{curl}; \Omega)$ -conforming  $W_h$  and the so-called bounded co-chain projection [3]. The bounded co-chain projection plays a critical role in the establishment of the Inf-Sup condition in the case of higher-order elements. For lower-order elements, a different argument without using this projection can be found in [37].

From [45, 49, 51, 41], we first recall the following  $\mu$ -weighted  $L^2$  orthogonal decomposition:

$$(4.39) \quad W_h = W_{0,h} + \nabla M_h,$$

where

$$(4.40) \quad W_{0,h} = \{\mathbf{w}_{0,h} \in W_h : (\mathbf{w}_{0,h}, \nabla \chi_h)_\mu = 0 \quad \forall \chi_h \in M_h\},$$

$$(4.41) \quad M_h = \{\chi_h \in H^1(\Omega)/\mathbb{R} : \chi_h|_T \in P_{l+1}(T) \quad \forall T \in \mathcal{T}_h\}.$$

THEOREM 4.3. *Under Assumption 1, we have the following Inf-Sup inequality:*

$$(4.42) \quad \sup_{\mathbf{0} \neq \mathbf{v}_h \in U_h} \frac{b(\mathbf{v}_h; (p_h, \mathbf{w}_{0,h}))}{\|\mathbf{v}_h\|_h} \geq C (\|p_h\|_0 + \|\mathbf{w}_{0,h}\|_0) \quad \forall p_h \in Q_h, \forall \mathbf{w}_{0,h} \in W_{0,h}.$$

Note that, corresponding to Theorem 4.2, the  $r$  in  $\|\cdot\|_h$  which is defined in (4.17) is replaced by  $r_0$ .

*Proof.* We have the  $\mu$ -weighted  $L^2$  orthogonal decomposition from Proposition 4.1:

$$(4.43) \quad \begin{aligned} \mathbf{w}_{0,h} &= \nabla q + \mathbf{w}^*, \quad q \in H^1(\Omega)/\mathbb{R}, \mathbf{w}^* \in H_0(\operatorname{div}^0 \mu; \Omega), \\ \|\mathbf{w}_{0,h}\|_{0,\mu}^2 &= \|\nabla q\|_{0,\mu}^2 + \|\mathbf{w}^*\|_{0,\mu}^2, \\ \|\mathbf{w}^*\|_{0,\mu} &\leq \|\mathbf{w}_{0,h}\|_{0,\mu}, \end{aligned}$$

where, from Assumption 1,

$$\begin{aligned} \mathbf{w}^* &\in H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega) \hookrightarrow \prod_{j=1}^J (H^r(\Omega_j))^3, \\ \sum_{j=1}^J \|\mathbf{w}^*\|_{r,\Omega_j} &\leq C \|\mathbf{curl} \mathbf{w}^*\|_0 = C \|\mathbf{curl} \mathbf{w}_{0,h}\|_0. \end{aligned}$$

Let  $\rho_h \mathbf{w}^* \in W_h$  denote the bounded co-chain projection of  $\mathbf{w}^* \in (L^2(\Omega))^3$  as constructed in [3], which satisfies

$$(4.44) \quad \rho_h \mathbf{w}^* \in W_h, \quad \|\rho_h \mathbf{w}^*\|_0 \leq C \|\mathbf{w}^*\|_0.$$

Moreover, for  $\nabla q$ , there exists a  $q_h \in M_h$  such that

$$(4.45) \quad \rho_h \nabla q = \nabla q_h.$$

We clearly have

$$\mathbf{w}_{0,h} = \rho_h \mathbf{w}_{0,h} = \rho_h \mathbf{w}^* + \nabla q_h,$$

but  $\mathbf{w}_{0,h} \in W_{0,h}$ , we have

$$\|\mathbf{w}_{0,h}\|_{0,\mu}^2 = (\rho_h \mathbf{w}^*, \mathbf{w}_{0,h})_\mu \leq C \|\rho_h \mathbf{w}^*\|_0 \|\mathbf{w}_{0,h}\|_{0,\mu} \leq C \|\mathbf{w}^*\|_0 \|\mathbf{w}_{0,h}\|_{0,\mu},$$

so we have

$$\|\mathbf{w}_{0,h}\|_{0,\mu} \leq C \|\mathbf{w}^*\|_0 \leq C \|\mathbf{w}^*\|_{0,\mu}.$$

Hence

$$(4.46) \quad C \|\mathbf{w}_{0,h}\|_{0,\mu} \leq \|\mathbf{w}^*\|_{0,\mu} \leq \|\mathbf{w}_{0,h}\|_{0,\mu}.$$

Consider the following problem: Given  $\mathbf{w}_{0,h} \in W_{0,h}, p_h \in Q_h$ , with  $\mathbf{w}^* \in H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega)$  being given by (4.43), find  $\mathbf{v}^* \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  such that

$$\mu^{-1} \mathbf{curl} \mathbf{v}^* = \mathbf{w}^*, \quad \operatorname{div} \varepsilon \mathbf{v}^* = p_h \quad \text{in } \Omega, \quad \mathbf{v}^* \times \mathbf{n}|_\Gamma = \mathbf{0}.$$

From Proposition 4.3 and Assumption 1 we know that the above problem has a unique solution  $\mathbf{v}^* \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega) \hookrightarrow \prod_{j=1}^J (H^r(\Omega_j))^3 \hookrightarrow (H^{r_0}(\Omega))^3$ , satisfying

$$(4.47) \quad \|\mathbf{v}^*\|_{r_0} \leq C \sum_{j=1}^J \|\mathbf{v}^*\|_{r,\Omega_j} \leq C \|\mathbf{v}^*\|_{\mathbf{curl}; \operatorname{div} \varepsilon} \leq C (\|p_h\|_0 + \|\mathbf{w}^*\|_0).$$

We also have

$$(4.48) \quad b(\mathbf{v}^*; (p_h, \mathbf{w}^*)) = (\mu^{-1} \mathbf{curl} \mathbf{v}^*, \mathbf{w}^*)_\mu + (\operatorname{div} \varepsilon \mathbf{v}^*, p_h) = \|p_h\|_0^2 + \|\mathbf{w}^*\|_{0,\mu}^2.$$



Let  $\mathbf{v}_h^* \in U_h$  be the Fortin-type interpolant to  $\mathbf{v}^* \in (H^{r_0}(\Omega))^3$ , as constructed in Theorem 4.2. We have

$$(4.49) \quad \begin{aligned} b(\mathbf{v}_h^*; (p_h, \mathbf{w}_{0,h})) &= b(\mathbf{v}_h^* - \mathbf{v}^*; (p_h, \mathbf{w}_{0,h})) + b(\mathbf{v}^*; (p_h, \mathbf{w}_{0,h})) \\ &= b(\mathbf{v}^*; (p_h, \mathbf{w}_{0,h})) \end{aligned}$$

$$(4.50) \quad \begin{aligned} &= b(\mathbf{v}^*; (p_h, \mathbf{w}^*)) + (\mathbf{v}^*; \mathbf{curl}(\mathbf{w}_{0,h} - \mathbf{w}^*)) \\ &= b(\mathbf{v}^*; (p_h, \mathbf{w}^*)) = \|p_h\|_0^2 + \|\mathbf{w}^*\|_{0,\mu}^2, \\ &\|\mathbf{v}_h^*\|_h \leq C\|\mathbf{v}^*\|_{\mathbf{curl};\text{div}\varepsilon} \leq C(\|p_h\|_0 + \|\mathbf{w}^*\|_{0,\mu}), \end{aligned}$$

$$(4.51) \quad \begin{aligned} \sup_{\mathbf{0} \neq \mathbf{v}_h \in U_h} \frac{b(\mathbf{v}_h; (p_h, \mathbf{w}_{0,h}))}{\|\mathbf{v}_h\|_h} &\geq \frac{b(\mathbf{v}_h^*; (p_h, \mathbf{w}_{0,h}))}{\|\mathbf{v}_h^*\|_h} \\ &\geq C(\|p_h\|_0 + \|\mathbf{w}^*\|_{0,\mu}), \end{aligned}$$

but  $\|\mathbf{w}^*\|_{0,\mu} \geq C\|\mathbf{w}_{0,h}\|_{0,\mu}$  from (4.46), and we complete the proof.  $\square$

Before closing this subsection, we point out two facts associated with  $\mathcal{K}_h(b)^\perp$ . These two facts will be used later on. We first note that the Inf-Sup inequality (4.42) holds over  $\mathcal{K}_h(b)^\perp$ , i.e.,

$$(4.52) \quad \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathcal{K}_h(b)^\perp} \frac{b(\mathbf{v}_h; (p_h, \mathbf{w}_{0,h}))}{\|\mathbf{v}_h\|_h} \geq C(\|p_h\|_0 + \|\mathbf{w}_{0,h}\|_{0,\mu}) \quad \forall p_h \in Q_h, \forall \mathbf{w}_{0,h} \in W_{0,h}.$$

We next note that the Fortin-type interpolation in Theorem 4.2 of

$$\mathbf{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\text{div}\varepsilon; \Omega) \hookrightarrow \prod_{j=1}^J (H^r(\Omega_j))^3 \hookrightarrow (H^{r_0}(\Omega))^3$$

can be taken in  $\mathcal{K}_h(b)^\perp$  only, i.e., the part  $\tilde{\mathbf{v}}_0^\perp$  of the  $L^2$  orthogonal decomposition of  $\tilde{\mathbf{v}} \in U_h$ ,

$$\tilde{\mathbf{v}} = \tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_0^\perp \quad \text{with } \tilde{\mathbf{v}}_0 \in \mathcal{K}_h(b) \text{ and } \tilde{\mathbf{v}}_0^\perp \in \mathcal{K}_h(b)^\perp.$$

We clearly have

$$(4.53) \quad b(\tilde{\mathbf{v}}_0^\perp; (p_h, \mathbf{w}_h)) = b(\mathbf{v}; (p_h, \mathbf{w}_h)) \quad \forall p_h \in Q_h, \forall \mathbf{w}_h \in W_h,$$

which can also be expressed in terms of  $L^2$  projectors  $R_h$  and  $\check{R}_h$  as follows:

$$(4.54) \quad R_h(\mu^{-1} \mathbf{curl}(\mathbf{v} - \tilde{\mathbf{v}}_0^\perp)) = \mathbf{0}, \quad \check{R}_h(\text{div}\varepsilon(\mathbf{v} - \tilde{\mathbf{v}}_0^\perp)) = 0.$$

From (4.24) and (4.34) we see that  $\tilde{\mathbf{v}}_0^\perp$  satisfies

$$(4.55) \quad \|\mathbf{v} - \tilde{\mathbf{v}}_0^\perp\|_0 \leq \|\mathbf{v} - \tilde{\mathbf{v}}\|_0 + \|\tilde{\mathbf{v}}_0\|_0 \leq Ch^{r_0} \|\mathbf{v}\|_{\mathbf{curl};\text{div}\varepsilon}.$$

By the inverse estimates on finite dimensional space  $U_h$

$$h^{r_0} \|\mathbf{v}_h\|_{r_0} \leq C \|\mathbf{v}_h\|_0 \quad \forall \mathbf{v}_h \in U_h,$$

we have from (4.25) and (4.34) that

$$(4.56) \quad \|\tilde{\mathbf{v}}_0^\perp\|_{r_0} \leq \|\tilde{\mathbf{v}}\|_{r_0} + \|\tilde{\mathbf{v}}_0\|_{r_0} \leq C \|\mathbf{v}\|_{\mathbf{curl};\text{div}\varepsilon} + h^{-r_0} \|\tilde{\mathbf{v}}_0\|_0 \leq C \|\mathbf{v}\|_{\mathbf{curl};\text{div}\varepsilon},$$

and we further have from Lemma 4.2, (4.22), (4.25), and (4.34)

$$(4.57) \quad \|\tilde{\mathbf{v}}_0^\perp\|_h \leq C \|\tilde{\mathbf{v}}\|_h \leq C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon},$$

$$(4.58) \quad \left( \sum_{K \in \mathcal{T}_h} h_K^{-2r_0} \|\mathbf{v} - \tilde{\mathbf{v}}_0^\perp\|_{0,K}^2 \right)^{1/2} + \|\tilde{\mathbf{v}}_0^\perp\|_{r_0} \leq C \|\mathbf{v}\|_{\mathbf{curl}; \text{div} \varepsilon}.$$

Thus, we have seen that the  $\tilde{\mathbf{v}}_0^\perp$  satisfies the same interpolation properties as  $\tilde{\mathbf{v}}$ , except that (4.23) does not hold for  $\tilde{\mathbf{v}}_0^\perp$ . Note that (4.23) are used to make (4.21) hold.

**4.3. The dual Fortin-type interpolation.** In this subsection, with the Inf-Sup inequality in (4.52), we show that there is a “dual” Fortin-type interpolation over  $Q_h \times W_h$  through the trilinear form  $b$ .

We consider the following problem: Given  $p \in H_0^1(\Omega)$  and  $\mathbf{w} \in H(\mathbf{curl}; \Omega) \cap H_0(\text{div}^0 \mu; \Omega)$ , find  $\tilde{p} \in Q_h$  and  $\tilde{\mathbf{w}} \in W_{0,h}$  such that

$$(4.59) \quad b(\mathbf{v}_h; (\tilde{p}, \tilde{\mathbf{w}})) = b(\mathbf{v}_h; (p, \mathbf{w})) \quad \forall \mathbf{v}_h \in \mathcal{X}_h(b)^\perp.$$

The choice of  $\mathcal{X}_h(b)^\perp$ , together with the established Inf-Sup inequality (4.52), ensures that the stated problem has a unique solution. Relative to the Fortin-type interpolation  $\tilde{\mathbf{v}}$  as constructed in Theorem 4.2, we call  $(\tilde{p}, \tilde{\mathbf{w}})$  the dual Fortin-type interpolation of  $(p, \mathbf{w})$ . For  $(q, \mathbf{z}) \in H^1(\Omega) \times (H^1(\Omega))^3$ , with  $r$  being in Assumption 1 and  $r_0 = \min(r, 1/2 - \epsilon)$ , corresponding to  $\|\cdot\|_h$  defined in (4.17) with  $r$  being replaced by  $r_0$ , we introduce

$$(4.60) \quad \begin{aligned} \|(q, \mathbf{z})\|_{*,h}^2 &:= \sum_{K \in \mathcal{T}_h} h_K^{2r_0-2} (\|q\|_{0,K}^2 + \|\mathbf{z}\|_{0,K}^2) \\ &+ \sum_{F \in \mathcal{F}_h^I} h_F^{2r_0-1} \int_F |\mathbf{n} \times (\mathbf{z} \times \mathbf{n})|^2 + \sum_{F \in \mathcal{F}_h^{\text{Inter}}} h_F^{2r_0-1} \int_F |q|^2. \end{aligned}$$

*Assumption 2.* In addition to Assumption 1, for any  $\mathbf{w} \in H(\mathbf{curl}; \Omega) \cap H_0(\text{div}^0 \mu; \Omega)$ , we assume that there exists a regular-singular decomposition as follows:

$$(4.61) \quad \mathbf{w} = \mathbf{w}^1 + \nabla p^1, \quad \mathbf{w}^1 \in (H^1(\Omega))^3, \quad p^1 \in H^1(\Omega) \cap \prod_{j=1}^J H^{1+r}(\Omega_j),$$

$$(4.62) \quad \|\mathbf{w}^1\|_1 + \|p^1\|_1 \leq C \|\mathbf{curl} \mathbf{w}\|_0, \quad \sum_{j=1}^J \|p^1\|_{1+r, \Omega_j} \leq C \|\mathbf{curl} \mathbf{w}\|_0.$$

If, additionally,  $\varepsilon^{-1} \mathbf{curl} \mathbf{w} \in H_0(\mathbf{curl}; \Omega) \cap H(\text{div}^0 \varepsilon; \Omega)$ , we further assume that

$$(4.63) \quad \mathbf{w}^1 \in \prod_{j=1}^J (H^{1+r}(\Omega_j))^3, \quad \sum_{j=1}^J \|\mathbf{w}^1\|_{1+r, \Omega_j} \leq C \|\mathbf{curl} \varepsilon^{-1} \mathbf{curl} \mathbf{w}\|_0.$$

The decomposition (4.61) is a well-known regular-singular decomposition in the literature, whatever  $\mu \in (W^{1,\infty}(\Omega))^{3 \times 3}$  or  $\mu$  as assumed in section 2, with  $\|\mathbf{w}^1\|_1 + \|p^1\|_1 \leq C \|\mathbf{curl} \mathbf{w}\|_0$ ; see [13, 14, 34, 42]. The second in (4.62) just follows from Assumption 1, with  $\mathbf{w} \in \prod_{j=1}^J (H^r(\Omega_j))^3$ . Regarding (4.63), it has been available only in the last decade; see [30, 34]. An intuitive observation of the regularity of  $\mathbf{w}^1$  in (4.63) is as follows. From Assumption 1 we have  $\mathbf{curl} \mathbf{w} \in \prod_{j=1}^J (H^r(\Omega_j))^3$ , so one

could expect  $\mathbf{w}^1 \in \prod_{j=1}^J (H^{1+r}(\Omega_j))^3$  due to  $\mathbf{curl} \mathbf{w} = \mathbf{curl} \mathbf{w}^1$ , with  $\mathbf{w}^1 \in (H^1(\Omega))^3$ . In two dimensions, this is exactly the case, since  $\mathbf{w}$  itself is of  $H^{1+r}$  function. Note that since  $r_0 = \min(r, 1/2 - \epsilon) \leq r$ , if Assumption 2 holds, then it also holds for  $r_0$  in place of  $r$  through (4.61) to (4.63).

**THEOREM 4.4.** *Assume that Assumption 1 holds. Let  $p \in H_0^1(\Omega)$ ,  $\mathbf{w} \in H(\mathbf{curl}; \Omega) \cap H_0(\text{div}^0 \mu; \Omega)$ , and let  $(\tilde{p}, \tilde{\mathbf{w}}) \in Q_h \times W_{0,h}$  be constructed as in problem (4.59). For any  $\mathbf{w}^1 \in (H^1(\Omega))^3$  which satisfies  $\mathbf{curl} \mathbf{w}^1 = \mathbf{curl} \mathbf{w}$  and for any  $\mathbf{z}_h^1 \in W_h^c = (V_h^1 \cap H^1(\Omega))^3 \subset W_h$ , choosing a  $\chi_h^1 \in M_h$  such that  $\mathbf{z}_h := \mathbf{z}_h^1 + \nabla \chi_h^1 \in W_{0,h}$ , for any  $q_h \in Q_h$ , we have*

$$(4.64) \quad \|\tilde{p} - q_h\|_0 + \|\tilde{\mathbf{w}} - \mathbf{z}_h\|_0 \leq C \|(p - q_h, \mathbf{w}^1 - \mathbf{z}_h^1)\|_{*,h},$$

$$(4.65) \quad \|\tilde{p}\|_0 + \|\tilde{\mathbf{w}}\|_0 \leq C(\|\nabla p\|_0 + \|\mathbf{curl} \mathbf{w}\|_0).$$

*Proof.* We only show (4.64), while (4.65) can be shown more easily. From the Inf-Sup inequality (4.52) we have

$$\|\tilde{p} - q_h\|_0 + \|\tilde{\mathbf{w}} - \mathbf{z}_h\|_0 \leq C \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathcal{K}_h(b)^\perp} \frac{b(\mathbf{v}_h; (\tilde{p} - q_h, \tilde{\mathbf{w}} - \mathbf{z}_h))}{\|\mathbf{v}_h\|_h},$$

where

$$\begin{aligned} b(\mathbf{v}_h; (\tilde{p} - q_h, \tilde{\mathbf{w}} - \mathbf{z}_h)) &= b(\mathbf{v}_h; (\tilde{p} - p, \tilde{\mathbf{w}} - \mathbf{w})) + b(\mathbf{v}_h; (p - q_h, \mathbf{w} - \mathbf{z}_h)) \\ &= b(\mathbf{v}_h; (p - q_h, \mathbf{w} - \mathbf{z}_h)), \\ b(\mathbf{v}_h; (p - q_h, \mathbf{w} - \mathbf{z}_h)) &= (\mathbf{v}_h, \mathbf{curl}(\mathbf{w}^1 - \mathbf{z}_h^1) - \varepsilon \nabla(p - q_h)), \\ (\mathbf{v}_h, \mathbf{curl}(\mathbf{w}^1 - \mathbf{z}_h^1)) &= (\mathbf{curl} \mathbf{v}_h, \mathbf{w}^1 - \mathbf{z}_h^1) \\ &\quad - \int_{\Gamma} \mathbf{v}_h \times \mathbf{n} \cdot (\mathbf{n} \times (\mathbf{w}^1 - \mathbf{z}_h^1) \times \mathbf{n}), \\ (\mathbf{v}_h, \varepsilon \nabla(p - q_h)) &= - \sum_{K \in \mathcal{T}_h} (\text{div} \varepsilon \mathbf{v}_h, p - q_h)_{0,K} \\ &\quad + \sum_{F \in \mathcal{F}_h^{\text{Inter}}} \int_F [\varepsilon \mathbf{v}_h \cdot \mathbf{n}](p - q_h). \end{aligned}$$

We have

$$\|\tilde{p} - q_h\|_0 + \|\tilde{\mathbf{w}} - \mathbf{z}_h\|_0 \leq C \|(p - q_h, \mathbf{w}^1 - \mathbf{z}_h^1)\|_{*,h}. \quad \square$$

**COROLLARY 4.1.** *For  $p$  and  $\mathbf{w}$  as in Theorem 4.4, under Assumptions 1 and 2,*

$$(4.66) \quad \|p - \tilde{p}\|_0 + \|\mathbf{w} - \tilde{\mathbf{w}}\|_0 \leq Ch^{r_0}(\|p\|_1 + \|\mathbf{curl} \mathbf{w}\|_0).$$

*Proof.* From Assumption 1 we know that

$$\mathbf{w} \in \prod_{j=1}^J (H^r(\Omega_j))^3, \quad \sum_{j=1}^J \|\mathbf{w}\|_{r, \Omega_j} \leq C \|\mathbf{curl} \mathbf{w}\|_0.$$

From Assumption 2, the regular-singular decomposition of  $\mathbf{w}$  is as follows:

$$\begin{aligned} \mathbf{w} &= \mathbf{w}^1 + \nabla p^1, \quad \mathbf{w}^1 \in (H^1(\Omega))^3, \quad p^1 \in H^1(\Omega) \cap \prod_{j=1}^J H^{1+r}(\Omega_j), \\ \|\mathbf{w}^1\|_1 + \|p^1\|_1 + \sum_{j=1}^J \|p^1\|_{1+r, \Omega_j} &\leq C \|\mathbf{curl} \mathbf{w}\|_0. \end{aligned}$$

Since  $W_h^c = (V_h^l \cap H^1(\Omega))^3 \subset W_h$ , we choose  $\mathbf{z}_h^1 \in W_h^c$  and  $q_h \in Q_h$  such that

(4.67)

$$\begin{aligned}
\|\mathbf{w}^1 - \mathbf{z}_h^1\|_0 &\leq Ch\|\mathbf{w}^1\|_1 \leq Ch\|\mathbf{curl}\mathbf{w}\|_0, \\
\|p - q_h\|_0 &\leq Ch\|p\|_1, \\
\sum_{K \in \mathcal{T}_h} h_K^{2r-2} (\|\mathbf{w}^1 - \mathbf{z}_h^1\|_{0,K}^2 + \|p - q_h\|_{0,K}^2) \\
&\leq Ch^{2r} (\|\mathbf{w}^1\|_1 + \|p\|_1)^2 \leq Ch^{2r} (\|\mathbf{curl}\mathbf{w}\|_0 + \|p\|_1)^2, \\
\sum_{F \in \mathcal{F}_h^\Gamma} h_F^{2r-1} \|\mathbf{w}^1 - \mathbf{z}_h^1\|_{0,F}^2 &\leq C \sum_{K \in \mathcal{T}_h} (h_K^{2r-2} \|\mathbf{w}^1 - \mathbf{w}_h^1\|_{0,K}^2 + h_K^{2r} \|\mathbf{w}^1 - \mathbf{w}_h^1\|_{1,K}^2) \\
&\leq Ch^{2r} \|\mathbf{w}^1\|_1^2 \leq Ch^{2r} \|\mathbf{curl}\mathbf{w}\|_0^2, \\
\sum_{F \in \mathcal{F}_h^{\text{Inter}}} h_F^{2r-1} \|p - q_h\|_{0,F}^2 &\leq C \sum_{K \in \mathcal{T}_h} (h_K^{2r-2} \|p - q_h\|_{0,K}^2 + h_K^{2r} \|p - q_h\|_{1,K}^2) \\
&\leq Ch^{2r} \|p\|_1^2.
\end{aligned}$$

All the above hold for  $r_0$  in place of  $r$ . Combining these, with  $r_0$  replacing  $r$ , we have

$$(4.68) \quad \|\tilde{p} - q_h\|_0 + \|\tilde{\mathbf{w}} - \mathbf{z}_h\|_0 \leq C\|(p - q_h, \mathbf{w}^1 - \mathbf{z}_h^1)\|_{*,h} \leq Ch^{r_0} (\|p\|_1 + \|\mathbf{curl}\mathbf{w}\|_0).$$

We choose

$$(4.69) \quad \chi_h^1 \in M_h, \quad (\nabla \chi_h^1, \nabla \chi)_\mu = -(\mathbf{z}_h^1, \nabla \chi)_\mu \quad \forall \chi \in M_h,$$

$$(4.70) \quad \mathbf{z}_h = \mathbf{z}_h^1 + \nabla \chi_h^1 \in W_{0,h}.$$

For all  $\chi \in M_h$ ,

$$\begin{aligned}
(\nabla(\chi_h^1 - \chi), \nabla(\chi_h^1 - \chi))_\mu &= -(\mathbf{z}_h^1, \nabla(\chi_h^1 - \chi))_\mu - (\nabla \chi, \nabla(\chi_h^1 - \chi))_\mu \\
&= (\mathbf{w}^1 - \mathbf{z}_h^1, \nabla(\chi_h^1 - \chi))_\mu - (\mathbf{w}^1 + \nabla p^1, \nabla(\chi_h^1 - \chi))_\mu + (\nabla(p^1 - \chi), \nabla(\chi_h^1 - \chi))_\mu,
\end{aligned}$$

but, from  $\mathbf{w} \in H(\mathbf{curl}; \Omega) \cap H_0(\text{div}^0; \Omega)$ ,

$$-(\mathbf{w}^1 + \nabla p^1, \nabla(\chi_h^1 - \chi))_\mu = -(\mathbf{w}, \nabla(\chi_h^1 - \chi))_\mu = 0,$$

we have

$$\|\nabla(\chi_h^1 - \chi)\|_{0,\mu} \leq C(\|\mathbf{w}^1 - \mathbf{z}_h^1\|_0 + \|\nabla(p^1 - \chi)\|_0),$$

and we have

$$\begin{aligned}
\|\nabla(\chi_h^1 - p^1)\|_0 &\leq \|\nabla(p^1 - \chi)\|_0 + \|\nabla(\chi_h^1 - \chi)\|_0 \\
&\leq C(\|\mathbf{w}^1 - \mathbf{z}_h^1\|_0 + \|\nabla(p^1 - \chi)\|_0).
\end{aligned}$$

From the finite element interpolation theory in [12, 27, 54], we can find a  $\chi \in M_h$  (cf. Lemma 6.3 in [38]) so that

$$\|\nabla(p^1 - \chi)\|_0 \leq Ch^r \sum_{j=1}^J \|p^1\|_{1+r, \Omega_j},$$

and we have

$$(4.71) \quad \|\nabla(\chi_h^1 - p^1)\|_0 \leq Ch\|\mathbf{w}^1\|_1 + Ch^r \sum_{j=1}^J \|p^1\|_{1+r, \Omega_j} \leq Ch^r \|\mathbf{curl} \mathbf{w}\|_0.$$

Combining (4.67) and (4.71), with  $\mathbf{z}_h = \mathbf{z}_h^1 + \nabla \chi_h^1 \in W_{0,h}$ , we have

$$(4.72) \quad \|\mathbf{w} - \mathbf{z}_h\|_0 \leq \|\mathbf{w}^1 - \mathbf{z}_h^1\|_0 + \|\nabla(p^1 - \chi_h^1)\|_0 \leq Ch^r \|\mathbf{curl} \mathbf{w}\|_0.$$

Therefore, by the triangle inequality and the fact that  $r_0 \leq r$ , we have

$$(4.73) \quad \begin{aligned} \|p - \tilde{p}\|_0 + \|\mathbf{w} - \tilde{\mathbf{w}}\|_0 &\leq \|p - q_h\|_0 + \|q_h - \tilde{p}\|_0 + \|\mathbf{w} - \mathbf{z}_h\|_0 \\ &+ \|\mathbf{z}_h - \tilde{\mathbf{w}}\|_0 \leq Ch^{r_0}(\|p\|_1 + \|\mathbf{curl} \mathbf{w}\|_0). \end{aligned} \quad \square$$

**COROLLARY 4.2.** *For  $p$  and  $\mathbf{w}$  as in Theorem 4.4, under Assumption 2 with the additional assumption on  $\mathbf{w}$  and  $\mathbf{w}^1 \in \prod_{j=1}^J (H^{1+r}(\Omega_j))^3$ , if  $p \in \prod_{j=1}^J H^{1+r}(\Omega_j)$ , then, with  $r_0 = \min(r, 1/2 - \epsilon)$ ,*

$$(4.74) \quad \begin{aligned} \|p - \tilde{p}\|_0 + \|\tilde{\mathbf{w}} - \mathbf{z}_h\|_0 &\leq Ch^{2r_0} \left( \sum_{j=1}^J \|p\|_{1+r, \Omega_j} + \|\mathbf{w}^1\|_{1+r, \Omega_j} \right) \\ &\leq Ch^{2r_0} \left( \sum_{j=1}^J \|p\|_{1+r, \Omega_j} + \|\mathbf{curl} \varepsilon^{-1} \mathbf{curl} \mathbf{w}\|_0 \right), \end{aligned}$$

and, for all  $\mathbf{v} \in H_0(\mathbf{curl}; \Omega)$ ,

$$(4.75) \quad \begin{aligned} |(\mathbf{w} - \tilde{\mathbf{w}}, \mathbf{curl} \mathbf{v})| &\leq Ch^{2r_0} \left( \sum_{j=1}^J \|p\|_{1+r, \Omega_j} + \|\mathbf{w}^1\|_{1+r, \Omega_j} \right) \|\mathbf{curl} \mathbf{v}\|_0 \\ &\leq Ch^{2r_0} \left( \sum_{j=1}^J \|p\|_{1+r, \Omega_j} + \|\mathbf{curl} \varepsilon^{-1} \mathbf{curl} \mathbf{w}\|_0 \right) \|\mathbf{curl} \mathbf{v}\|_0. \end{aligned}$$

*Proof.* From (4.64) and the finite element interpolation theory it is not difficult to have (4.74). For all  $\mathbf{v} \in H_0(\mathbf{curl}; \Omega)$ ,

$$\begin{aligned} (\mathbf{w} - \tilde{\mathbf{w}}, \mathbf{curl} \mathbf{v}) &= (\mathbf{w} - \mathbf{z}_h, \mathbf{curl} \mathbf{v}) + (\mathbf{z}_h - \tilde{\mathbf{w}}, \mathbf{curl} \mathbf{v}), \\ (\mathbf{w} - \mathbf{z}_h, \mathbf{curl} \mathbf{v}) &= (\mathbf{w}^1 - \mathbf{z}_h^1, \mathbf{curl} \mathbf{v}) \leq Ch^{1+r} \sum_{j=1}^J \|\mathbf{w}^1\|_{1+r, \Omega_j} \|\mathbf{curl} \mathbf{v}\|_0, \end{aligned}$$

where we have used  $(\nabla(p^1 - \chi_h^1), \mathbf{curl} \mathbf{v}) = 0$ , and  $h^{1+r} \leq h^{2r}$ , and we obtain (4.75), noting that  $r_0 \leq r$ .  $\square$

Different from  $p - \tilde{p}$ , we cannot have  $\|\mathbf{w} - \tilde{\mathbf{w}}\|_0 = \mathcal{O}(h^{2r_0})$ , in general. However, as will be seen in (5.61) later on, (4.75) is sufficient for our purpose.

**5. Coercivity and error estimates.** With the Fortin-type interpolations established in the previous section we are now in a position to investigate the coercivity property and to analyze the error estimates associated with the eigenproblem (3.12).

**5.1. Coercivity.** In this subsection we shall use the dual Fortin-type interpolation to establish the coercivity property of the curl/div part of the bilinear form in (3.13), that is,

$$(5.1) \quad \mathcal{L}_h^\perp(\mathbf{u}, \mathbf{v}) := (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}))_\mu + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v})).$$

Note that if either of  $\mathbf{u}$  and  $\mathbf{v}$  belongs to  $\mathcal{X}_h(b)$ , then  $\mathcal{L}_h^\perp(\mathbf{u}, \mathbf{v}) = 0$ .

**THEOREM 5.1.** *Assume that Assumption 1 holds. We have*

$$(5.2) \quad \mathcal{L}_h^\perp(v_h, v_h) = \|R_h(\mu^{-1} \mathbf{curl} v_h)\|_{0,\mu}^2 + \|\check{R}_h(\operatorname{div} \varepsilon v_h)\|_0^2 \geq C \|\mathbf{v}_h\|_0^2 \quad \forall \mathbf{v}_h \in \mathcal{X}_h(b)^\perp,$$

where  $\mathcal{X}_h(b)^\perp$  is given in (4.3).

*Proof.* From Proposition 4.1 we write  $\mathbf{v}_h$  as the following  $\varepsilon$ -weighted  $L^2$  orthogonal decomposition:

$$\mathbf{v}_h = \varepsilon^{-1} \mathbf{curl} \mathbf{w} - \nabla p \quad p \in H_0^1(\Omega), \quad \mathbf{w} \in H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega),$$

$$\|\mathbf{v}_h\|_{0,\varepsilon}^2 = \|\nabla p\|_{0,\varepsilon}^2 + \|\mathbf{curl} \mathbf{w}\|_{0,\varepsilon^{-1}}^2,$$

and let  $\tilde{p} \in Q_h$ ,  $\tilde{\mathbf{w}} \in W_{0,h}$  be the dual Fortin-type interpolations of  $p$ ,  $\mathbf{w}$ , respectively, i.e.,

$$b(\mathbf{v}_h; (\tilde{p}, \tilde{\mathbf{w}})) - b(\mathbf{v}_h; (p, \mathbf{w})) = 0 \quad \forall \mathbf{v}_h \in \mathcal{X}_h(b)^\perp,$$

such that

$$(5.3) \quad \|\tilde{p}\|_0 + \|\tilde{\mathbf{w}}\|_0 \leq C (\|p\|_1 + \|\mathbf{curl} \mathbf{w}\|_0) \leq C \|\mathbf{v}_h\|_{0,\varepsilon}.$$

Let  $\alpha > 0$  be a constant to be given. We have

$$\begin{aligned} \|R_h(\mu^{-1} \mathbf{curl} v_h)\|_{0,\mu}^2 &= \|R_h(\mu^{-1} \mathbf{curl} v_h) - \alpha \tilde{\mathbf{w}}\|_{0,\mu}^2 \\ &\quad + 2\alpha (R_h(\mu^{-1} \mathbf{curl} v_h), \tilde{\mathbf{w}})_\mu - \alpha^2 \|\tilde{\mathbf{w}}\|_{0,\mu}^2, \end{aligned}$$

where

$$\begin{aligned} (R_h(\mu^{-1} \mathbf{curl} v_h), \tilde{\mathbf{w}})_\mu &= (\mathbf{v}_h, \mathbf{curl} \tilde{\mathbf{w}}) = (\mathbf{v}_h, \mathbf{curl}(\tilde{\mathbf{w}} - \mathbf{w})) + (\mathbf{v}_h, \mathbf{curl} \mathbf{w}), \\ (\mathbf{v}_h, \mathbf{curl} \mathbf{w}) &= \|\mathbf{curl} \mathbf{w}\|_{0,\varepsilon^{-1}}^2, \end{aligned}$$

and we have

$$2\alpha (R_h(\mu^{-1} \mathbf{curl} v_h), \tilde{\mathbf{w}})_\mu = 2\alpha (\mathbf{v}_h, \mathbf{curl}(\tilde{\mathbf{w}} - \mathbf{w})) + 2\alpha \|\mathbf{curl} \mathbf{w}\|_{0,\varepsilon^{-1}}^2.$$

On the other hand, with the same  $\alpha$ , we have

$$\|\check{R}_h(\operatorname{div} \varepsilon v_h)\|_0^2 = \|\check{R}_h(\operatorname{div} \varepsilon v_h) - \alpha \tilde{p}\|_0^2 + 2\alpha (\check{R}_h(\operatorname{div} \varepsilon v_h), \tilde{p}) - \alpha^2 \|\tilde{p}\|_0^2,$$

where

$$\begin{aligned} (\check{R}_h(\operatorname{div} \varepsilon v_h), \tilde{p}) &= -(\mathbf{v}_h, \varepsilon \nabla \tilde{p}) = (\mathbf{v}_h, \varepsilon \nabla(p - \tilde{p})) - (\mathbf{v}_h, \varepsilon \nabla p), \\ -(\mathbf{v}_h, \varepsilon \nabla p) &= \|\nabla p\|_{0,\varepsilon}^2, \end{aligned}$$

and we have

$$2\alpha (\check{R}_h(\operatorname{div} \varepsilon v_h), \tilde{p}) = 2\alpha (\mathbf{v}_h, \varepsilon \nabla(p - \tilde{p})) + 2\alpha \|\nabla p\|_{0,\varepsilon}^2.$$

We therefore have

$$\begin{aligned} & \|R_h(\mu^{-1}\mathbf{curl}\mathbf{v}_h)\|_{0,\mu}^2 + \|\check{R}_h(\operatorname{div}\varepsilon\mathbf{v}_h)\|_0^2 \\ &= \|R_h(\mu^{-1}\mathbf{curl}\mathbf{v}_h) - \alpha\tilde{\mathbf{w}}\|_{0,\mu}^2 + \|\check{R}_h(\operatorname{div}\varepsilon\mathbf{v}_h) - \alpha\tilde{p}\|_0^2 \\ &+ 2\alpha((\mathbf{v}_h, \mathbf{curl}(\tilde{\mathbf{w}} - \mathbf{w})) + (\mathbf{v}_h, -\varepsilon\nabla(\tilde{p} - p))) \\ &+ 2\alpha(\|\nabla p\|_{0,\varepsilon}^2 + \|\mathbf{curl}\mathbf{w}\|_{0,\varepsilon^{-1}}^2) - \alpha^2(\|\tilde{p}\|_0^2 + \|\tilde{\mathbf{w}}\|_{0,\mu}^2), \end{aligned}$$

where

$$\begin{aligned} & (\mathbf{v}_h, \mathbf{curl}(\tilde{\mathbf{w}} - \mathbf{w})) + (\mathbf{v}_h, -\varepsilon\nabla(\tilde{p} - p)) = b(\mathbf{v}_h; (\tilde{p} - p, \tilde{\mathbf{w}} - \mathbf{w})) = 0, \\ & \|\nabla p\|_{0,\varepsilon}^2 + \|\mathbf{curl}\mathbf{w}\|_{0,\varepsilon^{-1}}^2 = \|\mathbf{v}_h\|_{0,\varepsilon}^2, \\ & \|\tilde{p}\|_0^2 + \|\tilde{\mathbf{w}}\|_{0,\mu}^2 \leq C\|\mathbf{v}_h\|_{0,\varepsilon}^2. \end{aligned}$$

Hence, by choosing a suitable  $\alpha > 0$ , we have for all  $\mathbf{v}_h \in \mathcal{K}_h(b)^\perp$

$$\|R_h(\mu^{-1}\mathbf{curl}\mathbf{v}_h)\|_{0,\mu}^2 + \|\check{R}_h(\operatorname{div}\varepsilon\mathbf{v}_h)\|_0^2 \geq \alpha(2 - \alpha C)\|\mathbf{v}_h\|_{0,\varepsilon}^2.$$

The proof is finished.  $\square$

**5.2. Regularity results.** In this subsection we review the regularity results associated with the eigenproblem (1.1).

Consider the general source problem: Find  $\mathbf{u} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div}\varepsilon; \Omega)$  such that

$$\begin{aligned} & \mathcal{B}(\mathbf{u}, \mathbf{v}) = (\mu^{-1}\mathbf{curl}\mathbf{u}, \mathbf{curl}\mathbf{v}) + (\operatorname{div}\varepsilon\mathbf{u}, \operatorname{div}\varepsilon\mathbf{v}) - \beta(\varepsilon\mathbf{u}, \mathbf{v}) = (\varepsilon\mathbf{f}, \mathbf{v}) \\ (5.4) \quad & \forall \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div}\varepsilon; \Omega), \end{aligned}$$

where  $\beta \neq 0$ . Note that  $\mathcal{L}(\mathbf{u}, \mathbf{v})$  defined in (2) satisfies  $\mathcal{L}(\mathbf{u}, \mathbf{v}) = \mathcal{B}(\mathbf{u}, \mathbf{v})$  with  $\beta = -1$ . The corresponding boundary value problem is

$$(5.5) \quad \mathbf{curl}\mu^{-1}\mathbf{curl}\mathbf{u} - \varepsilon\nabla\operatorname{div}\varepsilon\mathbf{u} - \beta\varepsilon\mathbf{u} = \varepsilon\mathbf{f} \quad \text{in } \Omega, \quad \mathbf{u} \times \mathbf{n} = \mathbf{0}, \quad \operatorname{div}\varepsilon\mathbf{u} = 0 \quad \text{on } \Gamma.$$

By the well-known Fredholm alternative theorem, with the compact operator  $\mathbb{T}$ , it is not difficult to show the following proposition; see [49].

**PROPOSITION 5.1.** *For any  $\beta \leq 0$  or any  $\beta > 0$  which is not an eigenvalue of the eigenproblem (1.1), we have*

$$(5.6) \quad \|\mathbf{u}\|_{\mathbf{curl}; \operatorname{div}\varepsilon} \leq C\|\mathbf{f}\|_0.$$

Let  $\mathbf{u} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div}\varepsilon; \Omega)$  be the exact solution of problem (5.4) for  $\mathbf{f} \in (L^2(\Omega))^3$ . Noticing that  $\theta := \operatorname{div}\varepsilon\mathbf{u}$  satisfies  $\langle \theta, \operatorname{div}\varepsilon\nabla q \rangle + \beta(\theta, q) = -(\varepsilon\mathbf{f}, \nabla q)$  for all  $q \in D(\Delta_\varepsilon^{\operatorname{Dir}})$  the Dirichlet domain of the operator  $\Delta_\varepsilon^{\operatorname{Dir}} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ ,  $q \mapsto \operatorname{div}\varepsilon\nabla q$ , following a similar argument in Theorem 1.2 or Theorem 7.1 in [30] (see also Proposition 4.1 in [14]), it can be shown that  $\operatorname{div}\varepsilon\mathbf{u} \in H_0^1(\Omega)$ . Consequently, from (5.5),  $\mu^{-1}\mathbf{curl}\mathbf{u} \in H(\mathbf{curl}; \Omega)$ .

Under Assumption 1, from Propositions 5.1 and 2.1, it is not difficult to have Proposition 5.2 below.

**PROPOSITION 5.2.** *Let  $\mathbf{u} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div}\varepsilon; \Omega)$  be the exact solution of problem (5.5). Then, under Assumption 1,  $\mathbf{u}, \mu^{-1}\mathbf{curl}\mathbf{u} \in \prod_{j=1}^J (H^r(\Omega_j))^3$ , satisfying*

$$(5.7) \quad \sum_{j=1}^J \|\mathbf{u}\|_{r, \Omega_j} + \|\mu^{-1}\mathbf{curl}\mathbf{u}\|_{r, \Omega_j} + \|\operatorname{div}\varepsilon\mathbf{u}\|_1 + \|\mathbf{curl}\mu^{-1}\mathbf{curl}\mathbf{u}\|_0 \leq C\|\mathbf{f}\|_0.$$

Let  $\mathbf{u}$  denote the solution of problem (5.5). We put

$$(5.8) \quad p := \operatorname{div} \varepsilon \mathbf{u} \in H_0^1(\Omega), \quad \mathbf{w} := \mu^{-1} \operatorname{curl} \mathbf{u} \in H(\operatorname{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega).$$

We see that if  $\mathbf{f} \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ , then  $p \in H_0^1(\Omega)$  and  $\mathbf{w} \in H(\operatorname{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega)$ , respectively, satisfy

$$(5.9) \quad -\operatorname{div} \varepsilon \nabla p - \beta p = \operatorname{div} \varepsilon \mathbf{f} \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma,$$

$$(5.10) \quad -\operatorname{curl} \varepsilon^{-1} \operatorname{curl} \mathbf{w} = \beta \operatorname{curl} \mathbf{u} + \operatorname{curl} \mathbf{f}, \quad \operatorname{div} \mu \mathbf{w} = 0 \quad \text{in } \Omega, \quad \mu \mathbf{w} \cdot \mathbf{n} = 0, \\ \varepsilon^{-1} \operatorname{curl} \mathbf{w} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

Under Assumptions 1 and 2, from Proposition 5.1, it is not difficult to have Proposition 5.3 below.

**PROPOSITION 5.3.** *Let  $\mathbf{u} \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  be the exact solution of problem (5.5). Assume that  $\mathbf{f} \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ . With  $p = \operatorname{div} \varepsilon \mathbf{u} \in H_0^1(\Omega)$  and  $\mathbf{w} = \mu^{-1} \operatorname{curl} \mathbf{u} \in H(\operatorname{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega)$ , under Assumptions 1 and 2, we have  $p \in \prod_{j=1}^J H^{1+r}(\Omega_j)$  and  $\mathbf{w} = \mathbf{w}^1 + \nabla p^1$ , with  $p^1 \in H^1(\Omega) \cap \prod_{j=1}^J H^{1+r}(\Omega_j)$  and  $\mathbf{w}^1 \in \prod_{j=1}^J (H^{1+r}(\Omega_j))^3$ ,*

$$(5.11) \quad \sum_{j=1}^J \|p\|_{1+r, \Omega_j} + \|\mathbf{w}^1\|_{1+r, \Omega_j} + \|\operatorname{curl} \mathbf{w}\|_{r, \Omega_j} + \|p^1\|_{1+r, \Omega_j} + \|p^1\|_1 \leq C \|\mathbf{f}\|_{\operatorname{curl}; \operatorname{div} \varepsilon}.$$

**5.3. Error estimates of source problem.** In this subsection we analyze the error bounds between the exact solution and the finite element solution for the general source problem (5.5). This mainly consists of the consistency error estimates from the  $L^2$  projections in Lemma 5.1 and the  $L^2$  error estimates in Theorem 5.2.

The finite element problem of (5.4) is to find  $\mathbf{u}_h \in U_h$  such that, for all  $\mathbf{v}_h \in U_h$ ,

$$(5.12) \quad \mathcal{B}_h(\mathbf{u}_h, \mathbf{v}_h) = (R_h(\mu^{-1} \operatorname{curl} \mathbf{u}_h), R_h(\mu^{-1} \operatorname{curl} \mathbf{v}_h))_\mu \\ + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}_h), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_h)) - \beta(\varepsilon \mathbf{u}_h, \mathbf{v}_h) = (\varepsilon \mathbf{f}, \mathbf{v}_h).$$

Note that  $\mathcal{B}_h$  is also well-defined over  $H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ .

**LEMMA 5.1.** *Let  $\mathbf{u} \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  be the exact solution of problem (5.5) and  $\mathbf{u}_h \in U_h$  the finite element problem (5.12), respectively. Let  $\tilde{p} \in Q_h$  and  $\tilde{\mathbf{w}} \in W_h$  be the dual Fortin-type interpolations of  $p = \operatorname{div} \varepsilon \mathbf{u} \in H_0^1(\Omega)$  and  $\mathbf{w} = \mu^{-1} \operatorname{curl} \mathbf{u} \in H(\operatorname{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega)$ , respectively. Then, under Assumption 1,*

$$(5.13) \quad \mathcal{B}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) = (\mathbf{w} - \tilde{\mathbf{w}}, R_h(\mu^{-1} \operatorname{curl} \mathbf{v}))_\mu + (p - \tilde{p}, \check{R}_h(\operatorname{div} \varepsilon \mathbf{v})) \quad \forall \mathbf{v} \in \mathcal{X}_h(b)^\perp.$$

*Proof.* With  $p = \operatorname{div} \varepsilon \mathbf{u} \in H_0^1(\Omega)$  and  $\mathbf{w} = \mu^{-1} \operatorname{curl} \mathbf{u} \in H(\operatorname{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega)$ , we have for all  $\mathbf{v} \in U_h$

$$\begin{aligned} \mathcal{B}_h(\mathbf{u}_h, \mathbf{v}) &= (\varepsilon \mathbf{f}, \mathbf{v}) = (\mathbf{v}, \operatorname{curl} \mathbf{w} - \varepsilon \nabla p) - \beta(\varepsilon \mathbf{u}, \mathbf{v}) = b(\mathbf{v}; (p, \mathbf{w})) - \beta(\varepsilon \mathbf{u}, \mathbf{v}), \\ \mathcal{B}_h(\mathbf{u}, \mathbf{v}) &= (R_h(\mu^{-1} \operatorname{curl} \mathbf{u}), R_h(\mu^{-1} \operatorname{curl} \mathbf{v}))_\mu + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v})) - \beta(\varepsilon \mathbf{u}, \mathbf{v}) \\ &= (\mathbf{u}, \operatorname{curl} R_h(\mu^{-1} \operatorname{curl} \mathbf{v})) - (\mathbf{u}, \varepsilon \nabla \check{R}_h(\operatorname{div} \varepsilon \mathbf{v})) - \beta(\varepsilon \mathbf{u}, \mathbf{v}) \\ &= (\mu^{-1} \operatorname{curl} \mathbf{u}, R_h(\mu^{-1} \operatorname{curl} \mathbf{v}))_\mu + (\operatorname{div} \varepsilon \mathbf{u}, \check{R}_h(\operatorname{div} \varepsilon \mathbf{v})) - \beta(\varepsilon \mathbf{u}, \mathbf{v}) \\ &= (\mathbf{w}, R_h(\mu^{-1} \operatorname{curl} \mathbf{v}))_\mu + (p, \check{R}_h(\operatorname{div} \varepsilon \mathbf{v})) - \beta(\varepsilon \mathbf{u}, \mathbf{v}), \end{aligned}$$



and we have

$$(5.14) \quad \mathcal{B}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) = (\mathbf{w}, R_h(\mu^{-1} \mathbf{curl} \mathbf{v}))_\mu + (p, \check{R}_h(\operatorname{div} \varepsilon \mathbf{v})) - b(\mathbf{v}; (p, \mathbf{w})).$$

Since  $\tilde{p} \in Q_h$ ,  $\tilde{\mathbf{w}} \in W_h$  are the dual Fortin-type interpolations of  $p$  and  $\mathbf{w}$ , respectively, i.e.,

$$(5.15) \quad (R_h(\mu^{-1} \mathbf{curl} \mathbf{v}), \tilde{\mathbf{w}})_\mu + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{v}), \tilde{p}) = b(\mathbf{v}; (\tilde{p}, \tilde{\mathbf{w}})) = b(\mathbf{v}; (p, \mathbf{w})) \quad \forall \mathbf{v} \in \mathcal{K}_h(b)^\perp,$$

we thus obtain (5.13) from the sum of (5.14) and (5.15).  $\square$

**COROLLARY 5.1.** *Under Assumptions 1 and 1, for  $\mathbf{w} \in H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega)$ , we have, for all  $\mathbf{v} \in \mathcal{K}_h(b)^\perp$ ,*

$$(5.16) \quad |\mathcal{B}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v})| \leq Ch^{r_0}(\|p\|_1 + \|\mathbf{curl} \mathbf{w}\|_0)(\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v})\|_{0,\mu} + \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{v})\|_0).$$

*Proof.* This is a simple consequence of Lemma 5.1 and Corollary 4.1.  $\square$

**COROLLARY 5.2.** *Let  $\mathbf{u} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  and  $\mathbf{u}_h \in U_h$  be the exact solution to problem (5.5) and the finite element solution to problem (5.12), respectively. Let  $\tilde{\mathbf{u}} \in U_h$  be the Fortin-type interpolation of  $\mathbf{u}$ . For  $\mathbf{v}_h = \tilde{\mathbf{u}} - \mathbf{u}_h = \mathbf{v}_{0,h} + \mathbf{v}_{0,h}^\perp$  with  $\mathbf{v}_{0,h} \in \mathcal{K}_h(b)$  and  $\mathbf{v}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp$ , under Assumptions 1 and 2, we have*

$$(5.17) \quad |\mathcal{B}_h(\mathbf{v}_h, \mathbf{v}_{0,h}^\perp)| \leq Ch^{r_0}(\|\mathbf{u}\|_{\mathbf{curl}; \operatorname{div} \varepsilon} + \|p\|_1 + \|\mathbf{curl} \mathbf{w}\|_0)(\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0).$$

*Proof.* Under Assumption 1,  $\mathbf{u} \in \prod_{j=1}^J (H^r(\Omega_j))^3 \hookrightarrow (H^{r_0}(\Omega))^3$  and  $\|\mathbf{u}\|_{r_0} \leq C \sum_{j=1}^J \|\mathbf{u}\|_{r, \Omega_j} \leq C \|\mathbf{u}\|_{\mathbf{curl}; \operatorname{div} \varepsilon}$ . We have

$$(5.18) \quad \mathcal{B}_h(\mathbf{v}_h, \mathbf{v}_{0,h}^\perp) = \mathcal{B}_h(\tilde{\mathbf{u}} - \mathbf{u}_h, \mathbf{v}_{0,h}^\perp) = \mathcal{B}_h(\tilde{\mathbf{u}} - \mathbf{u}, \mathbf{v}_{0,h}^\perp) + \mathcal{B}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_{0,h}^\perp),$$

where Corollary 5.1 leads to

$$|\mathcal{B}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_{0,h}^\perp)| \leq Ch^{r_0}(\|p\|_1 + \|\mathbf{curl} \mathbf{w}\|_0)(\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0),$$

while from (4.36), Theorem 4.2, and the coercivity in Theorem 5.1, we have

$$(5.19) \quad \begin{aligned} & |\mathcal{B}_h(\tilde{\mathbf{u}} - \mathbf{u}, \mathbf{v}_{0,h}^\perp)| \\ &= |(R_h(\mu^{-1} \mathbf{curl}(\tilde{\mathbf{u}} - \mathbf{u})), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu + (\check{R}_h(\operatorname{div} \varepsilon(\tilde{\mathbf{u}} - \mathbf{u})), \\ & \quad \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) - \beta(\varepsilon(\tilde{\mathbf{u}} - \mathbf{u}), \mathbf{v}_{0,h}^\perp)| \\ &= |-\beta(\varepsilon(\tilde{\mathbf{u}} - \mathbf{u}), \mathbf{v}_{0,h}^\perp)| \leq Ch^{r_0} \|\mathbf{u}\|_{\mathbf{curl}; \operatorname{div} \varepsilon} \|\mathbf{v}_{0,h}^\perp\|_0 \\ &\leq Ch^{r_0} \|\mathbf{u}\|_{\mathbf{curl}; \operatorname{div} \varepsilon} (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0). \end{aligned}$$

Hence, (5.2) holds.  $\square$

To turn to the error estimates, we first analyze the errors between the finite element solution and the finite element interpolation.

Let  $\mathbf{u}_h \in U_h$  be the finite element solution of (5.12), and let  $\tilde{\mathbf{u}} \in U_h$  be the Fortin-type interpolation of the exact solution  $\mathbf{u} \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  of problem (5.5).

Writing  $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_{0,h} + \tilde{\mathbf{u}}_{0,h}^\perp$  with  $\tilde{\mathbf{u}}_{0,h} \in \mathcal{K}_h(b)$ ,  $\tilde{\mathbf{u}}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp$  and  $\mathbf{u}_h = \mathbf{u}_{0,h} + \mathbf{u}_{0,h}^\perp$  with  $\mathbf{u}_{0,h} \in \mathcal{K}_h(b)$ ,  $\mathbf{u}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp$ , we introduce

$$(5.20) \quad \mathbf{v}_h := \tilde{\mathbf{u}} - \mathbf{u}_h = \mathbf{v}_{0,h} + \mathbf{v}_{0,h}^\perp$$

with  $\mathbf{v}_{0,h} = \tilde{\mathbf{u}}_{0,h} - \mathbf{u}_{0,h} \in \mathcal{K}_h(b)$ ,  $\mathbf{v}_{0,h}^\perp = \tilde{\mathbf{u}}_{0,h}^\perp - \mathbf{u}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp$ .

We shall estimate  $\mathbf{v}_{0,h}^\perp$  from the duality argument. Choosing the  $\mathbf{v}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp$ , we consider the following auxiliary problem: Find  $\mathbf{u}^* \in H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  such that

$$(5.21) \quad \mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u}^* - \varepsilon \nabla \operatorname{div} \varepsilon \mathbf{u}^* - \beta \varepsilon \mathbf{u}^* = \varepsilon \mathbf{v}_{0,h}^\perp \quad \text{in } \Omega, \quad \mathbf{u}^* \times \mathbf{n} = \mathbf{0}, \quad \operatorname{div} \varepsilon \mathbf{u}^* = 0 \quad \text{on } \Gamma.$$

Letting

$$(5.22) \quad p^* := \operatorname{div} \varepsilon \mathbf{u}^* \in H_0^1(\Omega), \quad \mathbf{w}^* := \mu^{-1} \mathbf{curl} \mathbf{u}^* \in H(\mathbf{curl}; \Omega) \cap H_0(\operatorname{div}^0 \mu; \Omega),$$

we have from Propositions 5.1 and 5.2

$$(5.23) \quad \sum_{j=1}^J \|\mathbf{u}^*\|_{r, \Omega_j} + \|\mathbf{w}^*\|_{r, \Omega_j} + \|\mathbf{u}^*\|_{\mathbf{curl}; \operatorname{div} \varepsilon} + \|p^*\|_1 + \|\mathbf{curl} \mathbf{w}^*\|_0 \leq C \|\mathbf{v}_{0,h}^\perp\|_0.$$

**THEOREM 5.2.** *Let  $\mathbf{v}_{0,h}^\perp$  be defined as in (5.20). Under Assumptions 1 and 2, for all  $h < h^*$  with  $h^* < 1$  sufficiently small we have*

$$(5.24) \quad \|\mathbf{v}_{0,h}^\perp\|_0 \leq Ch^{r_0} (\|\mathbf{u}\|_{\mathbf{curl}; \operatorname{div} \varepsilon} + \|\operatorname{div} \varepsilon \mathbf{u}\|_1 + \|\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u}\|_0),$$

$$(5.25) \quad \begin{aligned} & \|R_h(\mu^{-1} \mathbf{curl}(\mathbf{u} - \mathbf{u}_h))\|_{0, \mu} + \|\check{R}_h(\operatorname{div} \varepsilon(\mathbf{u} - \mathbf{u}_h))\|_0 \\ &= \|R_h(\mu^{-1} \mathbf{curl}(\mathbf{u} - \mathbf{u}_{0,h}^\perp))\|_{0, \mu} + \|\check{R}_h(\operatorname{div} \varepsilon(\mathbf{u} - \mathbf{u}_{0,h}^\perp))\|_0 \\ &\leq Ch^{r_0} (\|\mathbf{u}\|_{\mathbf{curl}; \operatorname{div} \varepsilon} + \|\operatorname{div} \varepsilon \mathbf{u}\|_1 + \|\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u}\|_0), \end{aligned}$$

where  $\mathbf{u}_h = \mathbf{u}_{0,h} + \mathbf{u}_{0,h}^\perp$  is the finite element solution of (5.12), with  $\mathbf{u}_{0,h} \in \mathcal{K}_h(b)$ ,  $\mathbf{u}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp$ , and  $r_0 = \min(r, 1/2 - \epsilon)$ .

*Proof.* From (5.21) we have

$$(5.26) \quad \begin{aligned} C \|\mathbf{v}_{0,h}^\perp\|_0^2 &\leq \|\mathbf{v}_{0,h}^\perp\|_{0, \varepsilon}^2 = (\mathbf{curl} \mathbf{w}^*, \mathbf{v}_{0,h}^\perp) - (\varepsilon \nabla p^*, \mathbf{v}_{0,h}^\perp) - \beta (\varepsilon \mathbf{u}^*, \mathbf{v}_{0,h}^\perp) \\ &= b(\mathbf{v}_{0,h}^\perp; (p^*, \mathbf{w}^*)) - \beta (\varepsilon \mathbf{u}^*, \mathbf{v}_{0,h}^\perp) = (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}^*), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu \\ &\quad + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}^*), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) - \beta (\varepsilon \mathbf{u}^*, \mathbf{v}_{0,h}^\perp) + b(\mathbf{v}_{0,h}^\perp; (p^*, \mathbf{w}^*)) \\ &\quad - (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}^*), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu - (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}^*), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) \\ &:= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}^*), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu \\ &\quad + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}^*), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) - \beta (\varepsilon \mathbf{u}^*, \mathbf{v}_{0,h}^\perp), \\ I_2 &= b(\mathbf{v}_{0,h}^\perp; (p^*, \mathbf{w}^*)) \\ &\quad - (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}^*), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu - (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}^*), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)). \end{aligned}$$

In the following we estimate  $I_1$  and  $I_2$ . The estimates are divided into two steps.

*Step 1:* Estimate  $I_1$ . Let  $\mathbf{u}_h^* \in U_h$  be the Fortin-type interpolation of  $\mathbf{u}^*$ , and write it as the following decomposition:

$$(5.27) \quad \mathbf{u}_h^* = \mathbf{u}_{0,h}^* + \mathbf{u}_{0,h}^{*\perp},$$

where  $\mathbf{u}_{0,h}^* \in \mathcal{K}_h(b)$ ,  $\mathbf{u}_{0,h}^{*\perp} \in \mathcal{K}_h(b)^\perp$ , satisfying the properties as stated in (4.36), (4.54), and Theorem 4.2. We have

$$(5.28) \quad \begin{aligned} I_1 &= (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}^*), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu \\ &\quad + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}^*), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) - \beta(\varepsilon \mathbf{u}^*, \mathbf{v}_{0,h}^\perp) \\ &= (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}^*), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}^*), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) \\ &\quad - \beta(\varepsilon \mathbf{u}_h^*, \mathbf{v}_{0,h}^\perp) - \beta(\varepsilon(\mathbf{u}^* - \mathbf{u}_h^*), \mathbf{v}_{0,h}^\perp) \\ &= (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}_{0,h}^{*\perp}), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_h))_\mu + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}_{0,h}^{*\perp}), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_h)) \\ &\quad - \beta(\varepsilon \mathbf{u}_{0,h}^{*\perp}, \mathbf{v}_h) - \beta(\varepsilon(\mathbf{u}^* - \mathbf{u}_h^*), \mathbf{v}_{0,h}^\perp) \\ &= (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}_{0,h}^{*\perp}), R_h(\mu^{-1} \mathbf{curl}(\mathbf{u} - \mathbf{u}_h)))_\mu \\ &\quad + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}_{0,h}^{*\perp}), \check{R}_h(\operatorname{div} \varepsilon(\mathbf{u} - \mathbf{u}_h))) \\ &\quad - \beta(\varepsilon \mathbf{u}_{0,h}^{*\perp}, \mathbf{u} - \mathbf{u}_h) - \beta(\varepsilon \mathbf{u}_{0,h}^{*\perp}, \tilde{\mathbf{u}} - \mathbf{u}) - \beta(\varepsilon(\mathbf{u}^* - \mathbf{u}_h^*), \mathbf{v}_{0,h}^\perp) \\ &= \mathcal{B}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_{0,h}^{*\perp}) - \beta(\varepsilon \mathbf{u}_{0,h}^{*\perp}, \tilde{\mathbf{u}} - \mathbf{u}) - \beta(\varepsilon(\mathbf{u}^* - \mathbf{u}_h^*), \mathbf{v}_{0,h}^\perp), \end{aligned}$$

where we have used  $\mathbf{v}_h = \tilde{\mathbf{u}} - \mathbf{u}_h = \tilde{\mathbf{u}} - \mathbf{u} + \mathbf{u} - \mathbf{u}_h$  and the Fortin-type interpolation

$$(5.29) \quad \tilde{\mathbf{u}} = \tilde{\mathbf{u}}_{0,h} + \tilde{\mathbf{u}}_{0,h}^\perp \quad \text{with } \tilde{\mathbf{u}}_{0,h} \in \mathcal{K}_h(b), \tilde{\mathbf{u}}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp$$

of the exact solution  $\mathbf{u}$ . Define  $\overline{\mathbf{u}_{0,h}^{*\perp}}$  the element-local  $L^2$  projection of  $\mathbf{u}_{0,h}^{*\perp}$  by

$$(5.30) \quad \overline{\mathbf{u}_{0,h}^{*\perp}}|_K = \frac{\int_K \mathbf{u}_{0,h}^{*\perp}}{|K|} \quad \forall K \in \mathcal{T}_h,$$

for which we have the interpolation property

$$(5.31) \quad \|\overline{\mathbf{u}_{0,h}^{*\perp}} - \mathbf{u}_{0,h}^{*\perp}\|_0 \leq Ch^{r_0} \|\mathbf{u}_{0,h}^{*\perp}\|_{r_0}.$$

But, since  $\mathbf{u}_h^* = \mathbf{u}_{0,h}^* + \mathbf{u}_{0,h}^{*\perp}$ , with  $\mathbf{u}_{0,h}^* \in \mathcal{K}_h(b)$ ,  $\mathbf{u}_{0,h}^{*\perp} \in \mathcal{K}_h(b)^\perp$ , is the Fortin-type interpolation of  $\mathbf{u}^*$ , we have from (4.56)

$$(5.32) \quad \|\mathbf{u}_{0,h}^{*\perp}\|_{r_0} \leq C \|\mathbf{u}^*\|_{\mathbf{curl}; \operatorname{div} \varepsilon}.$$

Hence

$$(5.33) \quad \|\overline{\mathbf{u}_{0,h}^{*\perp}} - \mathbf{u}_{0,h}^{*\perp}\|_0 \leq Ch^{r_0} \|\mathbf{u}_{0,h}^{*\perp}\|_{r_0} \leq Ch^{r_0} \|\mathbf{u}^*\|_{\mathbf{curl}; \operatorname{div} \varepsilon} \leq Ch^{r_0} \|\mathbf{v}_{0,h}^\perp\|_0,$$

and we have

$$(5.34) \quad -\beta(\varepsilon \mathbf{u}_{0,h}^{*\perp}, \tilde{\mathbf{u}} - \mathbf{u}) = -\beta(\varepsilon(\mathbf{u}_{0,h}^{*\perp} - \overline{\mathbf{u}_{0,h}^{*\perp}}), \tilde{\mathbf{u}} - \mathbf{u}) \leq Ch^{2r_0} \|\mathbf{v}_{0,h}^\perp\|_0 \|\mathbf{u}\|_{\mathbf{curl}; \operatorname{div} \varepsilon},$$

where, since  $\varepsilon \overline{\mathbf{u}_{0,h}^{*\perp}} \in \mathbf{P}_0(\varepsilon; K) \subset \mathbf{P}_{l-1}(\varepsilon; K)$ , we have used (4.23) and (4.25). We also have from (5.23)

$$(5.35) \quad -\beta(\varepsilon(\mathbf{u}^* - \mathbf{u}_h^*), \mathbf{v}_{0,h}^\perp) \leq Ch^{r_0} \|\mathbf{u}^*\|_{\mathbf{curl}; \operatorname{div} \varepsilon} \|\mathbf{v}_{0,h}^\perp\|_0 \leq Ch^{r_0} \|\mathbf{v}_{0,h}^\perp\|_0^2.$$

On the other hand, for  $\mathbf{u}_{0,h}^* \perp \in \mathcal{K}_h(b)^\perp$ , from Corollary 5.1, we have

$$(5.36) \quad |\mathcal{B}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_{0,h}^* \perp)| \leq C h^{r_0} (\|p\|_1 + \|\mathbf{curl} \mathbf{w}\|_0) (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{u}_{0,h}^* \perp)\|_{0,\mu} + \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}_{0,h}^* \perp)\|_0),$$

where

$$(5.37) \quad \begin{aligned} & \|R_h(\mu^{-1} \mathbf{curl} \mathbf{u}_{0,h}^* \perp)\|_{0,\mu} + \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}_{0,h}^* \perp)\|_0 \\ &= \|R_h(\mu^{-1} \mathbf{curl} \mathbf{u}_h^*)\|_{0,\mu} + \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}_h^*)\|_0 \\ &= \|R_h(\mu^{-1} \mathbf{curl} \mathbf{u}^*)\|_{0,\mu} + \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}^*)\|_0 \\ &\leq C(\|\mu^{-1} \mathbf{curl} \mathbf{u}^*\|_{0,\mu} + \|\operatorname{div} \varepsilon \mathbf{u}^*\|_0) \\ &\leq C\|\mathbf{v}_{0,h}^\perp\|_0. \end{aligned}$$

Therefore, from (14), (5.34), (5.35), (5.36), and (14) we have

$$(5.38) \quad |I_1| \leq C h^{r_0} \|\mathbf{v}_{0,h}^\perp\|_0^2 + C h^{2r_0} \|\mathbf{v}_{0,h}^\perp\|_0 \|\mathbf{u}\|_{\mathbf{curl}; \operatorname{div} \varepsilon} + C h^{r_0} (\|p\|_1 + \|\mathbf{curl} \mathbf{w}\|_0) \|\mathbf{v}_{0,h}^\perp\|_0.$$

*Step 2:* Estimate  $I_2$ . Letting  $\tilde{p}^* \in Q_h$ ,  $\tilde{\mathbf{w}}^* \in W_h$  be the dual Fortin-type interpolations of  $p^* = \operatorname{div} \varepsilon \mathbf{u}^*$ ,  $\mathbf{w}^* = \mu^{-1} \mathbf{curl} \mathbf{u}^*$ , respectively, i.e.,

$$(5.39) \quad b(\mathbf{v}; (p^*, \mathbf{w}^*)) = b(\mathbf{v}; (\tilde{p}^*, \tilde{\mathbf{w}}^*)) \quad \forall \mathbf{v} \in \mathcal{K}_h(b)^\perp,$$

we have

$$(5.40) \quad \begin{aligned} & (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}^*), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu + (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}^*), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) \\ &= (\mathbf{u}^*, \mathbf{curl} R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)) - (\mathbf{u}^*, \varepsilon \nabla \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) \\ &= (\mathbf{curl} \mathbf{u}^*, R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)) + (\operatorname{div} \varepsilon \mathbf{u}^*, \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) \\ &= (\mathbf{w}^*, R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu + (p^*, \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) \\ &= (\mathbf{w}^* - \tilde{\mathbf{w}}^*, R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu + (p^* - \tilde{p}^*, \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) \\ &\quad + (\tilde{\mathbf{w}}^*, R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu + (\tilde{p}^*, \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) \\ &= (\mathbf{w}^* - \tilde{\mathbf{w}}^*, R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu + (p^* - \tilde{p}^*, \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)) + b(\mathbf{v}_{0,h}^\perp; (\tilde{p}^*, \tilde{\mathbf{w}}^*)). \end{aligned}$$

We therefore obtain from (5.40), Corollary 4.1, and (5.23) that

$$(5.41) \quad \begin{aligned} |I_2| &= |b(\mathbf{v}_{0,h}^\perp; (p^*, \mathbf{w}^*)) - (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}^*), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu \\ &\quad - (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}^*), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp))| \\ &= |b(\mathbf{v}_{0,h}^\perp; (\tilde{p}^*, \tilde{\mathbf{w}}^*)) - (R_h(\mu^{-1} \mathbf{curl} \mathbf{u}^*), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu \\ &\quad - (\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}^*), \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp))| \\ &= |-(\mathbf{w}^* - \tilde{\mathbf{w}}^*, R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu - (p^* - \tilde{p}^*, \check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp))| \\ &\leq (\|\mathbf{w}^* - \tilde{\mathbf{w}}^*\|_0 + \|p^* - \tilde{p}^*\|_0) (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0) \\ &\leq C h^{r_0} (\|\mathbf{curl} \mathbf{w}^*\|_0 + \|p^*\|_1) (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0) \\ &\leq C h^{r_0} \|\mathbf{v}_{0,h}^\perp\|_0 (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0). \end{aligned}$$

Summarizing the estimates on  $I_1$  and  $I_2$  we have

$$(5.42) \quad (1 - Ch^{r_0}) \|\mathbf{v}_{0,h}^\perp\|_0^2 \leq Ch^{2r_0} \|\mathbf{v}_{0,h}^\perp\|_0 \|\mathbf{u}\|_{\mathbf{curl}; \text{div} \varepsilon} + Ch^{r_0} (\|p\|_1 + \|\mathbf{curl} \mathbf{w}\|_0) \|\mathbf{v}_{0,h}^\perp\|_0 \\ + Ch^{r_0} \|\mathbf{v}_{0,h}^\perp\|_0 (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\text{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0).$$

Taking

$$h < h^* < 1 \quad \text{with } h^{*r_0} \leq 1/C,$$

we have

$$(5.43) \quad \|\mathbf{v}_{0,h}^\perp\|_0 \leq Ch^{2r_0} \|\mathbf{u}\|_{\mathbf{curl}; \text{div} \varepsilon} + Ch^{r_0} (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\text{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0) \\ + Ch^{r_0} (\|p\|_1 + \|\mathbf{curl} \mathbf{w}\|_0),$$

that is,

$$(5.44) \quad \|\mathbf{v}_{0,h}^\perp\|_0 \leq Ch^{r_0} (\|\mathbf{u}\|_{\mathbf{curl}; \text{div} \varepsilon} + \|\text{div} \varepsilon \mathbf{u}\|_1 + \|\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u}\|_0) \\ + Ch^{r_0} (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\text{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0).$$

In what follows, we shall estimate  $\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\text{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0$ . We have

$$(5.45) \quad \|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu}^2 + \|\check{R}_h(\text{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0^2 \\ = \|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_h)\|_{0,\mu}^2 + \|\check{R}_h(\text{div} \varepsilon \mathbf{v}_h)\|_0^2 - \beta \|\mathbf{v}_{0,h}^\perp\|_{0,\varepsilon}^2 + \beta \|\mathbf{v}_{0,h}^\perp\|_{0,\varepsilon}^2 \\ = (R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_h), R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp))_\mu + (\check{R}_h(\text{div} \varepsilon \mathbf{v}_h), \check{R}_h(\text{div} \varepsilon \mathbf{v}_{0,h}^\perp)) \\ - \beta (\varepsilon \mathbf{v}_h, \mathbf{v}_{0,h}^\perp) + \beta \|\mathbf{v}_{0,h}^\perp\|_{0,\varepsilon}^2 = \mathcal{B}_h(\mathbf{v}_h, \mathbf{v}_{0,h}^\perp) + \beta \|\mathbf{v}_{0,h}^\perp\|_{0,\varepsilon}^2,$$

where, from Corollary 5.2,

$$(5.46) \quad |\mathcal{B}_h(\mathbf{v}_h, \mathbf{v}_{0,h}^\perp)| \leq Ch^{r_0} (\|\mathbf{u}\|_{\mathbf{curl}; \text{div} \varepsilon} + \|p\|_1 + \|\mathbf{curl} \mathbf{w}\|_0) (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} \\ + \|\check{R}_h(\text{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0) \\ = Ch^{r_0} (\|\mathbf{u}\|_{\mathbf{curl}; \text{div} \varepsilon} + \|\text{div} \varepsilon \mathbf{u}\|_1 + \|\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u}\|_0) \times \\ \times (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\text{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0).$$

Hence, we have from (5.44), (5.45), and (5.46) that for  $h < h^*$ ,

$$(5.47) \quad (1 - Ch^{*2r_0}) (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu}^2 + \|\check{R}_h(\text{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0^2) \\ \leq Ch^{2r_0} (\|\mathbf{u}\|_{\mathbf{curl}; \text{div} \varepsilon} + \|\text{div} \varepsilon \mathbf{u}\|_1 + \|\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u}\|_0)^2 \\ + Ch^{r_0} (\|\mathbf{u}\|_{\mathbf{curl}; \text{div} \varepsilon} + \|\text{div} \varepsilon \mathbf{u}\|_1 + \|\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u}\|_0) (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu} \\ + \|\check{R}_h(\text{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0),$$

from which we have, for some suitable small  $0 < \delta < 1 - Ch^{*2r_0}$ ,

$$(5.48) \quad (1 - Ch^{*2r_0} - \delta) (\|R_h(\mu^{-1} \mathbf{curl} \mathbf{v}_{0,h}^\perp)\|_{0,\mu}^2 + \|\check{R}_h(\text{div} \varepsilon \mathbf{v}_{0,h}^\perp)\|_0^2) \\ \leq Ch^{2r_0} (\|\mathbf{u}\|_{\mathbf{curl}; \text{div} \varepsilon} + \|\text{div} \varepsilon \mathbf{u}\|_1 + \|\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u}\|_0)^2,$$

that is to say, we have

$$(5.49) \quad \|R_h(\mu^{-1}\mathbf{curl}\mathbf{v}_{0,h}^\perp)\|_{0,\mu} + \|\check{R}_h(\operatorname{div}\varepsilon\mathbf{v}_{0,h}^\perp)\|_0 \leq Ch^{r_0}(\|\mathbf{u}\|_{\mathbf{curl};\operatorname{div}\varepsilon} + \|\operatorname{div}\varepsilon\mathbf{u}\|_1 + \|\mathbf{curl}\mu^{-1}\mathbf{curl}\mathbf{u}\|_0).$$

Therefore, we obtain (5.24) from (5.44) and (5.49),

$$(5.50) \quad \|\mathbf{v}_{0,h}^\perp\|_0 \leq Ch^{r_0}(\|\mathbf{u}\|_{\mathbf{curl};\operatorname{div}\varepsilon} + \|\operatorname{div}\varepsilon\mathbf{u}\|_1 + \|\mathbf{curl}\mu^{-1}\mathbf{curl}\mathbf{u}\|_0).$$

Moreover, from (5.49) and the interpolation property of  $\tilde{\mathbf{u}}_{0,h}^\perp$  we can have (5.25).  $\square$

From (5.24) and Proposition 5.2, we have

$$(5.51) \quad \|\mathbf{v}_{0,h}^\perp\|_0 \leq Ch^{r_0}\|\mathbf{f}\|_0.$$

If the right-hand side  $\mathbf{f}$  is more regular, we can obtain  $\mathcal{O}(h^{2r_0})$  error bound for  $\mathbf{v}_{0,h}^\perp$ . This result is stated in following lemma.

LEMMA 5.2. *Under the same assumptions as in Theorem 5.2,*

$$(5.52) \quad \|\mathbf{v}_{0,h}^\perp\|_0 \leq Ch^{2r_0}(\|\mathbf{u}\|_{\mathbf{curl};\operatorname{div}\varepsilon} + \|\operatorname{div}\varepsilon\mathbf{u}\|_1 + \|\mathbf{curl}\mu^{-1}\mathbf{curl}\mathbf{u}\|_0) + C \frac{(\mathbf{w} - \tilde{\mathbf{w}}, R_h(\mu^{-1}\mathbf{curl}\mathbf{u}_{0,h}^{\star\perp}))_\mu + (p - \tilde{p}, \check{R}_h(\operatorname{div}\varepsilon\mathbf{u}_{0,h}^{\star\perp}))}{\|\mathbf{v}_{0,h}^\perp\|_0},$$

$$(5.53) \quad \|\mathbf{v}_{0,h}^\perp\|_0 \leq Ch^{2r_0} \left( \|\mathbf{u}\|_{\mathbf{curl};\operatorname{div}\varepsilon} + \|p\|_1 + \|\mathbf{curl}\mathbf{w}\|_0 + \sum_{j=1}^J \|p\|_{1+r,\Omega_j} + \|\mathbf{w}^1\|_{1+r,\Omega_j} \right).$$

*Proof.* In fact, from (14), (5.34), (5.35), and (5.41), we have obtained (5.52). In what follows, we show (5.53). Under Assumption 2,  $\mathbf{w}^1 \in \prod_{j=1}^J (H^{1+r}(\Omega_j))^3$ , satisfying  $\mathbf{curl}\mathbf{w}^1 = \mathbf{curl}\mathbf{w}$ , and from Proposition 5.3 we know that  $p = \operatorname{div}\varepsilon\mathbf{u} \in \prod_{j=1}^J H^{1+r}(\Omega_j)$ . Instead of (5.36), we estimate  $\mathcal{B}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_{0,h}^{\star\perp})$  in a different way as follows:

$$(5.54) \quad \begin{aligned} \mathcal{B}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_{0,h}^{\star\perp}) &= (\mathbf{w} - \tilde{\mathbf{w}}, R_h(\mu^{-1}\mathbf{curl}\mathbf{u}_{0,h}^{\star\perp}))_\mu + (p - \tilde{p}, \check{R}_h(\operatorname{div}\varepsilon\mathbf{u}_{0,h}^{\star\perp})) \\ &= (\mathbf{w} - \tilde{\mathbf{w}}, R_h(\mu^{-1}\mathbf{curl}\mathbf{u}^*))_\mu + (p - \tilde{p}, \check{R}_h(\operatorname{div}\varepsilon\mathbf{u}^*)) \\ &= (\mathbf{w} - \tilde{\mathbf{w}}, R_h(\mu^{-1}\mathbf{curl}\mathbf{u}^*) - \mu^{-1}\mathbf{curl}\mathbf{u}^*)_\mu \\ &\quad + (p - \tilde{p}, \check{R}_h(\operatorname{div}\varepsilon\mathbf{u}^*) - \operatorname{div}\varepsilon\mathbf{u}^*) \\ &\quad + (\mathbf{w} - \tilde{\mathbf{w}}, \mathbf{curl}\mathbf{u}^*) + (p - \tilde{p}, \operatorname{div}\varepsilon\mathbf{u}^*). \end{aligned}$$

Since  $R_h(\mu^{-1}\mathbf{curl}\mathbf{u}^*)$  is  $\mu$ -weighted  $L^2$  projection of  $\mu^{-1}\mathbf{curl}\mathbf{u}^*$ , we have

$$(5.55) \quad \begin{aligned} \|R_h(\mu^{-1}\mathbf{curl}\mathbf{u}^*) - \mu^{-1}\mathbf{curl}\mathbf{u}^*\|_{0,\mu} &= \inf_{\mathbf{z}_h \in W_h} \|\mathbf{z}_h - \mu^{-1}\mathbf{curl}\mathbf{u}^*\|_{0,\mu} \\ &= \inf_{\mathbf{z}_h \in W_h} \|\mathbf{z}_h - \mathbf{w}^*\|_{0,\mu} \leq Ch^r \|\mathbf{v}_{0,h}^\perp\|_0, \end{aligned}$$

where we have chosen an  $H^1$ -conforming finite element interpolation  $\mathbf{z}_h \in W_h^c = (V_h^I \cap H^1(\Omega))^3 \subset W_h$  such that  $\|\mathbf{z}_h - \mathbf{w}^*\|_0 \leq Ch^r \sum_{j=1}^J \|\mathbf{w}^*\|_{r,\Omega_j} = \sum_{j=1}^J \|\mu^{-1}\mathbf{curl}\mathbf{u}^*\|_{r,\Omega_j} \leq Ch^r \|\mathbf{v}_{0,h}^\perp\|_0$ . Similarly, since  $\check{R}_h(\operatorname{div}\varepsilon\mathbf{u}^*)$  is  $L^2$  projection of  $\operatorname{div}\varepsilon\mathbf{u}^*$ , we have

$$(5.56) \quad \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}^*) - \operatorname{div} \varepsilon \mathbf{u}^*\|_0 = \inf_{q_h \in Q_h} \|q_h - \operatorname{div} \varepsilon \mathbf{u}^*\|_0 = \inf_{q_h \in Q_h} \|q_h - p^*\|_0 \\ \leq Ch \|p^*\|_1 = Ch \|\operatorname{div} \varepsilon \mathbf{u}^*\|_1 \leq Ch^r \|\mathbf{v}_{0,h}^\perp\|_0,$$

while, from Corollary 4.2, we have

$$(5.57) \quad |(\mathbf{w} - \tilde{\mathbf{w}}, \operatorname{curl} \mathbf{u}^*) + (p - \tilde{p}, \operatorname{div} \varepsilon \mathbf{u}^*)| \\ \leq Ch^{2r_0} \left( \sum_{j=1}^J \|\mathbf{w}^1\|_{1+r, \Omega_j} + \|p\|_{1+r, \Omega_j} \right) \|\mathbf{u}^*\|_{\operatorname{curl}; \operatorname{div} \varepsilon} \\ \leq Ch^{2r_0} \left( \sum_{j=1}^J \|\mathbf{w}^1\|_{1+r, \Omega_j} + \|p\|_{1+r, \Omega_j} \right) \|\mathbf{v}_{0,h}^\perp\|_0.$$

So we have

$$(5.58) \quad |\mathcal{B}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_{0,h}^{\perp})| \leq Ch^{2r_0} \left( \sum_{j=1}^J \|\mathbf{w}^1\|_{1+r, \Omega_j} + \|p\|_{1+r, \Omega_j} \right) \|\mathbf{v}_{0,h}^\perp\|_0. \quad \square$$

From Lemma 5.2, Theorem 5.2, and Propositions 5.2 and 5.3, we can have the following corollary.

**COROLLARY 5.3.** *Assume that Assumptions 1 and 2 hold. With the same  $\mathbf{v}_{0,h}^\perp$  as in Theorem 5.2, if  $\mathbf{f} \in (L^2(\Omega))^3$ , we have*

$$(5.59) \quad \|\mathbf{v}_{0,h}^\perp\|_0 \leq Ch^{r_0} \|\mathbf{f}\|_0.$$

As a result, we have

$$(5.60) \quad \|\mathbf{u} - \mathbf{u}_{0,h}^\perp\|_0 + \|\mathbf{u} - \mathbf{u}_{0,h}^\perp\|_{\mathcal{L}_h^\perp} \leq Ch^{r_0} \|\mathbf{f}\|_0,$$

where  $\|\mathbf{v}\|_{\mathcal{L}_h^\perp}^2 := \mathcal{L}_h^\perp(\mathbf{v}, \mathbf{v})$  as defined in (5.1). If  $\mathbf{f} \in \cap H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ , then

$$(5.61) \quad \|\mathbf{v}_{0,h}^\perp\|_0 \leq Ch^{2r_0} \|\mathbf{f}\|_{\operatorname{curl}; \operatorname{div} \varepsilon}.$$

**COROLLARY 5.4.** *The finite element problem (5.12) has a unique solution.*

*Proof.* We follow the argument in [53]. Note that uniqueness and existence are equivalent for a finite dimensional square system. We need only consider the uniqueness. If the solution of (5.12) is not unique, then for  $\mathbf{f} = \mathbf{0}$ , there should be a solution  $\mathbf{u}_h \neq \mathbf{0}$ . For such  $\mathbf{u}_h = \mathbf{u}_{0,h} + \mathbf{u}_{0,h}^\perp$  with  $\mathbf{u}_{0,h} \in \mathcal{K}_h(b)$ ,  $\mathbf{u}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp$ , from the error estimates (5.60) we know that  $\mathbf{u}_{0,h}^\perp = \mathbf{0}$ , since  $\mathbf{f} = \mathbf{0}$  leads to  $\mathbf{u} = \mathbf{0}$ . Meanwhile, noting that  $\mathbf{u}_{0,h}$  is determined by

$$-\beta(\varepsilon \mathbf{u}_{0,h}, \mathbf{v}) = (\varepsilon \mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{K}_h(b), \quad \square$$

but  $\mathbf{f} = \mathbf{0}$ , we have  $\mathbf{u}_{0,h} = \mathbf{0}$ . Hence  $\mathbf{u}_h = \mathbf{0}$ , which contradicts  $\mathbf{u}_h \neq \mathbf{0}$ .

From Corollary 5.3 and Theorem 4.1, it is not difficult to show the following corollary.

**COROLLARY 5.5.** *Let  $\mathbf{u}_h = \mathbf{u}_{0,h} + \mathbf{u}_{0,h}^\perp$  with  $\mathbf{u}_{0,h} \in \mathcal{K}_h(b)$ ,  $\mathbf{u}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp$  be the solution to the finite element problem (5.12). Let  $\mathbf{u} \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$  be the solution of problem (5.5). If  $\mathbf{f} \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ , then, under Assumptions 1 and 2, we have*

$$\|\mathbf{u}_{0,h}\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 \leq Ch^{r_0} \|\mathbf{f}\|_{\operatorname{curl}; \operatorname{div} \varepsilon}.$$

Note that  $\mathcal{L}_h(\cdot, \cdot) = \mathcal{B}_h(\cdot, \cdot)$  with  $\beta = -1$ . Define an energy norm

$$\|\mathbf{v}\|_{\mathcal{L}_h}^2 := \|\mathbf{v}\|_{0,\varepsilon}^2 + \|R_h(\mu^{-1}\mathbf{curl}\mathbf{v})\|_{0,\mu}^2 + \|\check{R}_h(\operatorname{div}\varepsilon\mathbf{v})\|_0^2.$$

From Corollaries 5.3 and 5.5, we have established the following error estimates:

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{L}_h} \leq C h^{r_0} \|\mathbf{f}\|_{\mathbf{curl};\operatorname{div}\varepsilon}.$$

**6. Error estimates for the eigenproblem.** In this section, we shall analyze the error estimates of the eigenproblem (2.4) and the finite element eigenproblem (3.12), or, equivalently, the error estimates of the eigenpairs of the operators  $\mathbb{T}$  and  $\mathbb{T}_h$  which are, respectively, defined in (2.8) and (3.14). We first investigate the well-posedness of the eigenproblem (3.12) and review the error estimates of the source problem in terms of the operators  $\mathbb{T}$  and  $\mathbb{T}_h$ .

**THEOREM 6.1.** *The eigenproblem (3.12) is well-posed, and  $\lambda_h = 1$  is the only one whose eigenfunction space is  $\mathcal{K}_h(b)$ . For any other eigenvalues  $\lambda_h \neq 1$ , it satisfies*

$$(6.1) \quad \lambda_h \geq 1 + C > 1,$$

and its eigenfunction space belongs to  $\mathcal{K}_h(b)^\perp$ , where  $C$  comes from Theorem 5.1. Moreover, for different eigenvalues, their eigenfunctions are orthogonal in both the  $\mathcal{L}_h$ -induced inner product  $(\cdot, \cdot)_{\mathcal{L}_h} = \mathcal{L}_h(\cdot, \cdot)$  and the  $\varepsilon$ -weighted  $L^2$  inner product  $(\cdot, \cdot)_\varepsilon$ .

*Proof.* The eigenproblem (3.12) is well-posed, thanks to the coercivity of  $\mathcal{L}_h$  on  $U_h$ , i.e.,

$$(6.2) \quad \mathcal{L}_h(\mathbf{v}_h, \mathbf{v}_h) = \|R_h(\mu^{-1}\mathbf{curl}\mathbf{v}_h)\|_{0,\mu}^2 + \|\check{R}_h(\operatorname{div}\varepsilon\mathbf{v}_h)\|_0^2 + \|\mathbf{v}_h\|_{0,\varepsilon}^2 \geq \|\mathbf{v}_h\|_{0,\varepsilon}^2 > 0 \\ \forall \mathbf{v}_h (\neq \mathbf{0}) \in U_h.$$

We first note that all the eigenvalues are real numbers, because of the symmetry property of (3.12). From (4.4) and (4.6), we can verify the orthogonality of eigenfunctions corresponding to different eigenvalues with respect to both  $(\cdot, \cdot)_{\mathcal{L}_h}$  and  $(\cdot, \cdot)_\varepsilon$ . Also, we can see that  $\lambda_h = 1$  is an eigenvalue and its eigenspace is  $\mathcal{K}_h(b)$ . For any other eigenvalue  $\lambda_h \neq 1$ , with eigenfunction  $\mathbf{u}_h \in U_h$ , we have (6.1) and

$$(6.3) \quad \mathbf{u}_h \in \mathcal{K}_h(b)^\perp.$$

In fact, write

$$\mathbf{u}_h = \mathbf{u}_{0,h} + \mathbf{u}_{0,h}^\perp, \quad \text{where } \mathbf{u}_{0,h} \in \mathcal{K}_h(b) \text{ and } \mathbf{u}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp.$$

If  $\mathbf{u}_{0,h} = \mathbf{0}$ , then (6.3) holds. Otherwise, from  $\mathcal{L}_h(\mathbf{u}_h, \mathbf{v}) = (\varepsilon\mathbf{u}_h, \mathbf{v}) = (\varepsilon\mathbf{u}_{0,h}, \mathbf{v})$  for all  $\mathbf{v} \in \mathcal{K}_h(b)$ , we have

$$(\varepsilon\mathbf{u}_{0,h}, \mathbf{v}) = \lambda_h (\varepsilon\mathbf{u}_{0,h}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{K}_h(b).$$

Thus, we find that both  $(\varepsilon\mathbf{u}_{0,h}, \mathbf{u}_{0,h}) = \lambda_h (\varepsilon\mathbf{u}_{0,h}, \mathbf{u}_{0,h})$  and  $\lambda_h \neq 1$  lead to  $\|\mathbf{u}_{0,h}\|_{0,\varepsilon} = 0$ , i.e.,  $\mathbf{u}_{0,h} = \mathbf{0}$ . Hence,  $\mathbf{u}_h = \mathbf{u}_{0,h}^\perp \in \mathcal{K}_h(b)^\perp$ , i.e., (6.3) holds. Since  $\mathbf{u}_h \in \mathcal{K}_h(b)^\perp$ ,  $\mathbf{u}_h \neq \mathbf{0}$ , we have from Theorem 5.1 that  $\|\mathbf{u}_h\|_{\mathcal{L}_h^\perp}^2 \geq C\|\mathbf{u}_h\|_{0,\varepsilon}^2 > 0$ . From (3.12), (3.13), and Theorem 5.1, we then have  $\lambda_h \|\mathbf{u}_h\|_{0,\varepsilon}^2 = \|\mathbf{u}_h\|_{\mathcal{L}_h}^2 = \|\mathbf{u}_h\|_{\mathcal{L}_h^\perp}^2 + \|\mathbf{u}_h\|_{0,\varepsilon}^2 \geq (1+C)\|\mathbf{u}_h\|_{0,\varepsilon}^2$ , which yields (6.1).  $\square$



According to (2.5), the only a priori known eigenvalue  $\lambda_h = 1$  should be abandoned, whose eigenfunction space is  $\mathcal{K}_h(b)$ . Such a situation also exists in other methods [23, 22]. In terms of  $\mathbb{T}_h$ ,  $\nu_h = 1$  is the only a priori known eigenvalue with eigenspace  $\mathcal{K}_h(b)$ , and it follows that all the other eigenvalues of  $\mathbb{T}_h$  satisfy

$$(6.4) \quad 0 < \nu_h \leq 1/(1+C) < 1$$

and the associated eigenspaces belong to  $\mathcal{K}_h(b)^\perp$ . From (6.4), there exists a fixed gap  $\gamma := 1/(1+C) < 1$  independent of  $h$  such that all the eigenvalues  $\nu_h < 1$  and the only a priori known eigenvalue  $\nu_h = 1$  are well-separated.

By decomposing  $\mathbb{T}_h \mathbf{f}$ , which is determined as in (3.14), as

$$(6.5) \quad \mathbb{T}_h \mathbf{f} = \mathbb{T}_{0,h} \mathbf{f} + \mathbb{T}_{0,h}^\perp \mathbf{f}, \quad \text{where } \mathbb{T}_{0,h} \mathbf{f} \in \mathcal{K}_h(b) \text{ and } \mathbb{T}_{0,h}^\perp \mathbf{f} \in \mathcal{K}_h(b)^\perp,$$

we know that any eigenpair  $(\lambda_h \neq 1, \mathbf{u}_h \neq \mathbf{0})$  satisfies (3.12) if and only if

$$(6.6) \quad \mathbb{T}_{0,h}^\perp \mathbf{u}_h = \nu_h \mathbf{u}_h \quad \text{with } \nu_h = 1/\lambda_h \text{ and } \mathbf{u}_h \in \mathcal{K}_h(b)^\perp.$$

In the following, we will analyze the errors between eigenpairs of (6.6) and (2.9). Note that  $\mathcal{L}_h = \mathcal{B}_h$  with  $\beta = -1$ . We recall (5.60) in Corollary 5.3 in terms of  $\mathbb{T}_{0,h}^\perp$  in the following:

$$(6.7) \quad \|\mathbb{T} \mathbf{f} - \mathbb{T}_{0,h}^\perp \mathbf{f}\|_0 \leq Ch^{r_0} \|\mathbf{f}\|_0 \quad \forall \mathbf{f} \in (L^2(\Omega))^3,$$

$$(6.8) \quad \|\mathbb{T} \mathbf{f} - \mathbb{T}_{0,h}^\perp \mathbf{f}\|_{\mathcal{L}_h} \leq Ch^{r_0} \|\mathbf{f}\|_0 \leq Ch^{r_0} \|\mathbf{f}\|_{\mathcal{L}_h} \quad \forall \mathbf{f} \in (L^2(\Omega))^3.$$

In what follows, we shall show the spectral correctness property and the optimal error bound of the finite element method (3.12), in the spirit of [10], since the underlying  $\mathbb{T}$  is self-adjoint and compact. We recall the following theorem on spectral correctness stated in [50], which is due to [10, 52].

**THEOREM 6.2.** *Let  $X$  be a Hilbert space with inner product  $(\cdot, \cdot)_X$  and norm  $\|\cdot\|_X$  and  $A : X \rightarrow X$  a self-adjoint and compact operator. Let  $\Theta = \{h_n \in \mathbb{R} : 1 \leq n < \infty\}$  be a discrete subset such that  $h_n \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $A_h$  which is defined with respect to  $h \in \Theta$  denote a family of linear self-adjoint operators. Assume that  $A_h$  converges pointwise to  $A$  and that the set  $\mathcal{A} = \{A_h : X \rightarrow X, h \in \Theta\}$  is collectively compact. Let  $\nu$  be an eigenvalue of  $A$  of multiplicity  $m$  and let  $\phi_i$ ,  $1 \leq i \leq m$ , be the associated orthonormal eigenvectors. Then, (a) for any  $\rho > 0$  such that the disk  $B(\nu, \rho)$  which centers at  $\nu$  with radius  $\rho$  contains no other eigenvalues of  $A$ , there exists  $h_\rho$  which depends on  $\rho$  such that for all  $h < h_\rho$ ,  $A_h$  has exactly  $m$  eigenvalues (repeated according to their multiplicity) in the disk  $B(\nu, \rho)$ ; (b) for  $h < h_\rho$ , denoting the set of the eigenvalues of  $A_h$  in the disk  $B(\nu, \rho)$  as  $\nu_{h,i}$ ,  $1 \leq i \leq m$ , we have, for all  $1 \leq i \leq m$ ,*

$$(6.9) \quad c|\nu - \nu_{h,i}| \leq \sum_{k,l=1}^m |((A - A_h)\phi_k, \phi_l)_X| + \sum_{l=1}^m \|(A - A_h)\phi_l\|_X^2.$$

Put  $X := (L^2(\Omega))^3$ ,  $(\cdot, \cdot)_X := (\cdot, \cdot)_\varepsilon$  and  $\|\cdot\|_X := \|\cdot\|_{0,\varepsilon}$ ,  $A := \mathbb{T}$ ,  $\Theta := \{h : h = \max_{K \in \mathcal{T}_h} h_K\}$ , and  $A_h := \mathbb{T}_{0,h}^\perp$ . As mentioned in section 2,  $\mathbb{T}$  is self-adjoint and compact from  $(L^2(\Omega))^3$  to  $(L^2(\Omega))^3$ . It can be easily verified that  $\mathbb{T}_{0,h}^\perp : (L^2(\Omega))^3 \rightarrow$

$(L^2(\Omega))^3$  is also self-adjoint. The error estimates in (6.7) ensure the pointwise convergence of  $\mathbb{T}_{0,h}^\perp$  to  $\mathbb{T}$  and the collective compactness of the sequence  $\{\mathbb{T}_{0,h}^\perp\}_{h>0}$ . Therefore, Theorem 6.2 holds.

From the error estimates in operator norm in (6.7), it is not difficult to obtain the error estimates for eigenvalues within the general spectral theory for compact operators in [10].

**THEOREM 6.3.** *Let  $\nu^{-1} = \lambda$  denote the eigenvalue of the eigenproblem (2.4) and  $\nu_h^{-1} = \lambda_h$  the eigenvalue of the finite element eigenproblem (3.12). Under Assumptions 1 and 2, with the global regularity  $r_0$  as defined after Assumption 1, we have*

$$|\nu - \nu_h| \leq Ch^{2r_0}.$$

In what follows, we shall briefly address how to deal with the Maxwell eigenproblem (1.2). For convenience, here we restate (1.2) as follows: Find  $(\omega_M^2, \mathbf{u}_M \neq \mathbf{0})$  such that

$$(6.10) \quad \mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u}_M = \omega_M^2 \varepsilon \mathbf{u}_M, \quad \operatorname{div} \varepsilon \mathbf{u}_M = 0 \quad \text{in } \Omega, \quad \mathbf{u}_M \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

Above and below, the subscript  $M$  is used to indicate that  $(\omega_M^2, \mathbf{u}_M)$  denotes the eigenpairs (i.e., eigenvalue and eigenfunction) of the Maxwell's eigenproblem. Associated with (6.10), the curlcurl-graddiv eigenproblem reads as follows: Find  $(\omega^2, \mathbf{u} \neq \mathbf{0})$  such that

$$(6.11) \quad \mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u} - s \varepsilon \nabla \operatorname{div} \varepsilon \mathbf{u} = \omega^2 \varepsilon \mathbf{u} \quad \text{in } \Omega, \quad \mathbf{u} \times \mathbf{n} = \mathbf{0}, \quad \operatorname{div} \varepsilon \mathbf{u} = 0 \quad \text{on } \Gamma,$$

where, comparing with (1.1), we see that there is an additional parameter  $s > 0$  which is referred to as the regularization parameter, whose role will be seen later on.

The eigenpairs of the eigenproblem (6.11) can be divided into two families; cf. [32, 29]. One family is  $(\omega_L^2, \mathbf{u}_L)$ , such that

$$(6.12) \quad \mathbf{curl} \mathbf{u}_L = \mathbf{0},$$

where  $(\omega_L^2, \mathbf{u}_L) := (s\kappa, \nabla \varphi) \in \mathbb{R} \times H_0(\mathbf{curl}; \Omega) \cap H(\operatorname{div} \varepsilon; \Omega)$ , and  $(\kappa, \varphi \neq 0) \in \mathbb{R} \times H_0^1(\Omega)$  denotes the eigenpair of the Laplace Dirichlet eigenproblem, such that

$$(6.13) \quad -\operatorname{div} \varepsilon \nabla \varphi = \kappa \varphi \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on } \Gamma.$$

Above and below, the subscript  $L$  is used to indicate that  $(\omega_L^2, \mathbf{u}_L)$  denotes the eigenpair related to the Laplace Dirichlet eigenproblem. The other is  $(\omega_M^2, \mathbf{u}_M)$ , the eigenpairs of the Maxwell's eigenproblem (6.10), such that

$$(6.14) \quad \operatorname{div} \varepsilon \mathbf{u}_M = 0.$$

All the eigenfunctions of (6.11) are independent of  $s$  (only their multiplicities depend on  $s$ ). Further, the Maxwell eigenvalues  $\omega_M^2$  do not depend on  $s$ , while the eigenvalues  $\omega_L^2$  depend on  $s$  linearly, i.e.,

$$(6.15) \quad \omega_L^2 = s\kappa.$$

Likewise, accordingly, the finite element eigenproblem of (6.11) will produce two families of eigenpairs. Those finite element eigenvalues  $\omega_h^2$  ( $\omega_h^2 := \lambda_h - 1$ ) which do not vary with  $s$  are approximations of Maxwell eigenvalues, while those  $\omega_h^2$  which vary linearly with  $s$  are not.

A practical approach is to solve the finite element eigenproblem with different values of the regularization parameter  $s$ . Let  $s$  increase, for example. Observe that those eigenvalues that are constant and do not vary when the value of  $s$  increases are Maxwell eigenvalues, while those that linearly vary with  $s$  are not. On the other hand, we can always set  $s = 1$ . Recall that (6.12) and (6.14) can be used to determine which eigenvalues are Maxwell eigenvalues and which are not. In fact, we define a quantity for a finite element eigenfunction  $\mathbf{u}_h$  as follows:

$$(6.16) \quad \tau(\mathbf{u}_h) := \frac{\|R_h(\mu^{-1} \mathbf{curl} \mathbf{u}_h)\|_{0,\mu}^2}{s \|\check{R}_h(\operatorname{div} \varepsilon \mathbf{u}_h)\|_0^2}.$$

According to the value of  $\tau(\mathbf{u}_h)$ , we can distinguish if  $\mathbf{u}_h$  is an approximation of Maxwell eigenfunctions or not. In other words, from (6.16), we can conclude that those  $\mathbf{u}_h$  with large  $\tau(\mathbf{u}_h)$  are Maxwell eigenfunctions, and otherwise not. These two approaches are used in [32, 31].

**7. Concluding remarks.** In this paper, we have presented a new  $H^1$ -conforming finite element method for the curlcurl-graddiv eigenvalue interface problem, where  $L^2$  projections are applied to the curl and the div operators. The  $H^1$ -conforming finite element space of the solution is the standard nodal-continuous Lagrange element, enriched with some element bubbles. The method is analyzed for the general source problem and the eigenproblem in a three-dimensional simply connected Lipschitz domain with connected boundary. Discontinuous and anisotropic, nonhomogeneous coefficients are allowed, which may lead to very low piecewise regularity in the solution. It is shown that the method is suitable for a non- $H^1$  space solution of piecewise  $H^r$  regularity, where  $r$  can be any number in  $[0, 1]$ . The error bounds relative to the piecewise  $H^r$  regularity of the solution have been established, where, particularly, the error bounds  $\mathcal{O}(h^{r_0})$  and  $\mathcal{O}(h^{2r_0})$  are obtained for eigenfunctions and eigenvalues, respectively, where  $r_0$  comes from the continuous embedding of the piecewise  $H^r$  into the global  $H^{r_0}$ . The spectrally correct property of the proposed finite element method is also shown.

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